

Unitarily invariant norms related to factors

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Abstract

Let \mathcal{M} be a semi-finite factor and let $\mathcal{J}(\mathcal{M})$ be the set of operators T in \mathcal{M} such that $T = ETE$ for some finite projection E . In this paper we obtain a representation theorem for unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ in terms of Ky Fan norms. As an application, we prove that the class of unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ coincides with the class of symmetric gauge norms on a classical abelian algebra, which generalizes von Neumann's classical result [28] on unitarily invariant norms on $M_n(\mathbb{C})$. As another application, Ky Fan's dominance theorem [7] is obtained for semi-finite factors. Some classical results in non-commutative L^p -theory (e.g., non-commutative Hölder's inequality, duality and reflexivity of non-commutative L^p -spaces) are extended to general unitarily invariant norms related to semi-finite factors. We also prove that up to a scale the operator norm is the unique unitarily invariant norm associated to a type III factor.

Keywords: semi-finite factors, unitarily invariant norms, s -numbers, Ky Fan norms.

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1 Introduction

F.J. Murray and J.von Neumann [15, 16, 17, 26, 27] introduced and studied certain algebras of Hilbert space operators. Those algebras are now called “Von Neumann algebras.” They are strong-operator closed self-adjoint subalgebras of all bounded linear transformations on a Hilbert space. *Factors* are von Neumann algebras whose centers consist of scalar multiples of the identity operator. Every von Neumann algebra is a direct sum (or “direct integral”) of factors. Thus factors are the building blocks for general von Neumann algebras. Murray and von Neumann [15] classified factors into type $I_n, I_\infty, II_1, II_\infty, III$ factors. Type I_n and I_∞ factors are full matrix algebras: $M_n(\mathbb{C})$ and $\mathcal{B}(\mathcal{H})$. Type I_n and II_1 factors are called finite factors. There is a unique faithful normal tracial state on a finite factor.

Factors except type III factors are called semi-finite factors. A semi-finite factor admits a faithful normal tracial weight.

The unitarily invariant norms on type I_n factors were introduced by von Neumann [28] for the purpose of metrizing matrix spaces. Von Neumann, together with his associates, established that the class of unitarily invariant norms on type I_n factors coincides with the class of symmetric gauge norms on \mathbb{C}^n . These norms have now been variously generalized and utilized in several contexts. For example, Schatten [21, 22] defined unitarily invariant norms on two-sided ideals of completely continuous operators in type I_∞ factors; Ky Fan [7] studied Ky Fan norms and obtained his dominance theorem. The unitarily invariant norms play a crucial role in the study of function spaces and group representations (see e.g. [12]) and in obtaining certain bounds of importance in quantum field theory (see [24]). For historical perspectives and surveys of unitarily invariant norms, see Schatten [21, 22], Hewitt and Ross [11], Gohberg and Krein [9] and Simon [24].

In [4], a structure theorem for unitarily invariant norms on finite factors is obtained. The main purpose of this paper is to set up a structure theorem for unitarily invariant norms related to arbitrary factors, which has a number of applications.

In this paper, a semi-finite von Neumann algebra (\mathcal{M}, τ) means a von Neumann algebra \mathcal{M} with a faithful normal tracial weight τ , and a Hilbert space \mathcal{H} means the separable infinite-dimensional complex Hilbert space. If (\mathcal{M}, τ) is a finite von Neumann algebra, we assume that $\tau(1) = 1$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we assume that $\tau = \text{Tr}$, the classical tracial weight on $\mathcal{B}(\mathcal{H})$. This paper is organized as the following.

In section 2, we define the s -numbers of operators in a semi-finite von Neumann algebra (\mathcal{M}, τ) from the point of view of non-increasing rearrangements of functions.

In section 3, we study various norms related to a semi-finite von Neumann algebra (\mathcal{M}, τ) . Let $\mathcal{J}(\mathcal{M})$ be the set of operators T in \mathcal{M} such that $T = ETE$ for some finite projection E . Then $\mathcal{J}(\mathcal{M})$ is a hereditary self-adjoint two-sided ideal of \mathcal{M} . If \mathcal{M} is a finite von Neumann algebra, then $\mathcal{J}(\mathcal{M}) = \mathcal{M}$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we simply write $\mathcal{J}(\mathcal{H})$ instead of $\mathcal{J}(\mathcal{B}(\mathcal{H}))$. Note that $\mathcal{J}(\mathcal{H})$ is the set of bounded linear operators T on \mathcal{H} such that both T and T^* are finite rank operators. A *unitarily invariant* norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is a norm on $\mathcal{J}(\mathcal{M})$ satisfying $\|UTV\| = \|T\|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators U, V in \mathcal{M} . For a semi-finite von Neumann algebra (\mathcal{M}, τ) , let $\text{Aut}(\mathcal{M}, \tau)$ be the set of $*$ -automorphisms on \mathcal{M} preserving τ . A *symmetric gauge* norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is a norm on $\mathcal{J}(\mathcal{M})$ such that $\|T\| = \||T|\|$ (gauge invariant) and $\|\theta(T)\| = \|T\|$ (symmetric) for all operators $T \in \mathcal{J}(\mathcal{M})$ and $\theta \in \text{Aut}(\mathcal{M}, \tau)$. A norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is a *normalized norm* if

$\|E\| = 1$ for a projection E in \mathcal{M} such that $\tau(E) = 1$. We will reserve the notation $\|\cdot\|$ for the operator norm on a von Neumann algebra.

In section 4, we define and study the normalized Ky Fan norms related to semi-finite von Neumann algebras. To illustrate difficulties one may encounter in studying of the unitarily invariant norms related to infinite factors, we point out here one example. The following result plays a key role in the studying of unitarily invariant norms on finite factors: if $\|\cdot\|$ is a normalized unitarily invariant norm on a finite factor (\mathcal{M}, τ) , then

$$\|T\|_1 \leq \|T\| \leq \|T\|$$

for all $T \in \mathcal{M}$, where $\|T\|_1 = \tau(|T|)$ (see Corollary 3.34 of [4]). However, the above result is not true for infinite factors (see Proposition 4.6).

In section 5, we study the dual norms of symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$. For $T \in \mathcal{J}(\mathcal{M})$, define

$$\|T\|^\# = \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}), \|X\| \leq 1\}.$$

In this section, we also compute the dual norms of Ky Fan norms and prove that $\|\cdot\|^{\#\#} = \|\cdot\|$ under certain conditions.

A representation theorem (Theorem 6.4) for symmetric gauge norms on $\mathcal{J}(\mathcal{M})$ is set up in section 6, which is the main result of this paper. In the rest sections of this paper, we give a number of applications of the representation theorem.

In section 7, we set up a representation theorem for unitarily invariant norms related to semi-finite factors and representation theorems for symmetric gauge norms related to the classical abelian von Neumann algebras $l^\infty(\mathbf{N})$ and $L^\infty[0, \infty)$.

In section 8, we prove that there is a one-to-one correspondence between unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ for a semi-finite factor \mathcal{M} and symmetric gauge norms on $\mathcal{J}(\mathcal{A})$ for a classical abelian von Neumann algebra \mathcal{A} , which generalizes von Neumann's classical result [28] on unitarily invariant norms on type I_n factors. Furthermore, we establish the one to one correspondence between the dual norms on $\mathcal{J}(\mathcal{M})$ for a semi-finite factor \mathcal{M} and the dual norms on $\mathcal{J}(\mathcal{A})$, which plays a key role in the studying of duality and reflexivity of the completion of $\mathcal{J}(\mathcal{M})$ with respect to unitarily invariant norms.

Ky Fan's dominance theorem for semi-finite factors is proved in section 9.

Let \mathcal{M} be an infinite semi-finite factor and $\|\cdot\|$ be a unitarily invariant norm on \mathcal{M} . We denote by $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ the completion of $\mathcal{J}(\mathcal{M})$ with respect to $\|\cdot\|$. Let $\widetilde{\mathcal{M}}$ be the completion of \mathcal{M} with respect to the measure topology in the sense of Nelson [18]. In section 10, we prove that there is an injective map from $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ into $\widetilde{\mathcal{M}}$ that extends the identity map from $\mathcal{J}(\mathcal{M})$ onto $\mathcal{J}(\mathcal{M})$. The reader should compare the proof of Proposition 10.10 and the proof of Proposition 12.3 of [4] to see the big difference between two cases: type II_1 case and type II_∞ case. An element in $\widetilde{\mathcal{M}}$ can be identified with a closed, densely defined operator affiliated with \mathcal{M} (see [18]). So generally speaking, an element in $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ should be treated as an unbounded operator. In section 10, we also extend the non-commutative Hölder's inequality to the general unitarily invariant norms.

In section 11, we study the duality and reflexivity of $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ for infinite semi-finite factors. We point out another example of difference between two cases: finite factors and infinite semi-finite factors. Let \mathcal{N} be a type II_1 factor with a faithful normal state $\tau_{\mathcal{N}}$. Let $\|\cdot\|$ be a unitarily invariant norm on \mathcal{N} and let $\|\cdot\|^\#$ be the dual unitarily invariant norm on \mathcal{N} . The following result is proved in [4]: $\overline{\mathcal{N}}_{\|\cdot\|^\#}$ is the dual space of $\overline{\mathcal{N}}_{\|\cdot\|}$ if and only if $\|\cdot\|$ is a continuous norm on \mathcal{N} , i.e., $\lim_{\tau(E) \rightarrow 0+} \|E\| = 0$. A key step to proving the above result is based on the following fact: if $\|\cdot\|$ is a continuous unitarily invariant norm on \mathcal{N} and $\phi \in \overline{\mathcal{N}}_{\|\cdot\|}^\#$, then the restriction of ϕ to \mathcal{N} is an ultraweakly continuous linear functional, i.e., ϕ is in the predual space of \mathcal{N} . However, it is easy to see that the similar result is not true for infinite semi-finite factors \mathcal{M} , e.g., $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

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2 Preliminaries

2.1 Nonincreasing rearrangements of functions

Throughout this paper, we denote by m the Lebesgue measure on $[0, \infty)$. In the following, a measurable function and a measurable set mean a Lebesgue measurable function and a Lebesgue measurable set, respectively. Let $f(x)$ be a real measurable function on $[0, \infty)$. The *nonincreasing rearrangement function*, $f^*(x)$, of $f(x)$ is defined by

$$f^*(x) = \sup\{y : m(\{f > y\}) > x\}, \quad 0 \leq x < \infty. \quad (1)$$

We summarize some well-known properties of $f^*(x)$ in the following proposition.

Proposition 2.1. *Let $f(x), g(x), f_1(x), f_2(x), \dots$ be real measurable functions on $[0, \infty)$, c be a real number. Then we have the following:*

1. $f^*(x)$ is a nonincreasing, right-continuous function on $[0, \infty)$ such that $f^*(0) = \text{esssup} f(x)$;
2. $(f + c)^* = f^* + c$;
3. $(cf)^* = cf^*$ if $c \geq 0$;
4. if $f(x)$ is a simple function, then $f^*(x)$ is also a simple function;
5. if $f(x) \leq g(x)$ for almost all x , then $f^*(x) \leq g^*(x)$ everywhere;
6. $\|f^*(x) - g^*(x)\|_\infty \leq \|f(x) - g(x)\|_\infty$;
7. if $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ uniformly, then $\lim_{n \rightarrow \infty} f_n^*(x) = f^*(x)$ uniformly;
8. if $f_n(x)$ converges to $f(x)$ in measure, then $\liminf_{n \rightarrow \infty} f_n^*(x) \geq f^*(x)$ for every $x \in [0, \infty)$;
9. if $f_n(x)$ converges to $f(x)$ in measure, then $\limsup_{n \rightarrow \infty} f_n^*(x) \leq f^*(x)$ for every $x \in [0, \infty)$ such that f^* is continuous at x ;
10. $f(x)$ and $f^*(x)$ are equi-measurable, i.e., for any real number y , $m(\{f > y\}) = m(\{f^* > y\})$;
11. $f^* = g^*$ if and only if $f(x)$ and $g(x)$ are equi-measurable;
12. if $f(x)$ and $g(x)$ are bounded functions and $\int_0^\infty f^n(x)dx = \int_0^\infty g^n(x)dx$ for all $n = 0, 1, 2, \dots$, then $f^*(x) = g^*(x)$;
13. $\int_0^\infty f(x)dx = \int_0^\infty f^*(x)dx$ when either integral is well-defined.

2.2 s -numbers of operators in type II_∞ factors

In [6], Fack and Kosaki give a rather complete exposition of generalized s -numbers of τ -measurable operators affiliated with semi-finite von Neumann algebras. For the reader's convenience and our purpose, we provide sufficient details on s -numbers of bounded operators in semi-finite von Neumann algebras in this section. We will define s -numbers of bounded operators in semi-finite von Neumann algebras from the point of view of non-increasing rearrangements of functions. The following lemma is well-known.

Lemma 2.2. *Let (\mathcal{A}, τ) be a separable (i.e., with separable predual) diffuse abelian von Neumann algebra with a faithful normal tracial weight τ on \mathcal{A} such that $\tau(1) = \infty$. Then there is a $*$ -isomorphism α from (\mathcal{A}, τ) onto $(L^\infty[0, \infty), \int_0^\infty dx)$ such that $\tau = \int_0^\infty dx \cdot \alpha$.*

Let \mathcal{M} be a type II_∞ factor and let τ be a faithful normal tracial weight on \mathcal{M} . For $T \in \mathcal{M}$, there is a separable diffuse abelian von Neumann subalgebra \mathcal{A} of \mathcal{M} containing $|T|$. By Lemma 2.2, there is a $*$ -isomorphism α from (\mathcal{A}, τ) onto $(L^\infty[0, \infty), \int_0^\infty dx)$ such that $\tau = \int_0^\infty dx \cdot \alpha$. Let $f(x) = \alpha(|T|)$ and let $f^*(x)$ be the non-increasing rearrangement of $f(x)$ (see (1)). Then the s -numbers of T , $\mu_s(T)$, are defined as

$$\mu_s(T) = f^*(s), \quad 0 \leq s < \infty.$$

Lemma 2.3. $\mu_s(T)$ does not depend on \mathcal{A} and α .

Proof. Let \mathcal{A}_1 be another separable diffuse abelian von Neumann subalgebra of \mathcal{M} containing $|T|$ and suppose β is a $*$ -isomorphism from \mathcal{A}_1 onto $L^\infty[0, \infty)$ such that $\tau = \int_0^\infty dx \cdot \beta$. Let $g(x) = \beta(|T|)$. For every number $n = 0, 1, 2, \dots$, $\int_0^\infty f^n(x) dx = \tau(|T|^n) = \int_0^\infty g^n(x) dx$. Since both $f(x)$ and $g(x)$ are bounded positive functions, by 12 of Proposition 2.1, $f^*(x) = g^*(x)$ for all $x \in [0, \infty)$. \square

Corollary 2.4. For $T \in \mathcal{M}$ and $p \geq 0$, $\tau(|T|^p) = \int_0^\infty \mu_s(T)^p ds$.

Lemma 2.5. Let E, F be two projections in \mathcal{M} . If $\tau(E^\perp) < \tau(F^\perp) < \infty$, then $\tau(E \wedge F^\perp) > 0$.

Proof. By Proposition 2.5.14 of [6] (page 119, vol 1), $R(F^\perp E^\perp) = F^\perp - E \wedge F^\perp$, where $R(F^\perp E^\perp)$ is the range projection of $F^\perp E^\perp$. Therefore,

$$\tau(E \wedge F^\perp) = \tau(F^\perp) - \tau(R(F^\perp E^\perp)) \geq \tau(F^\perp) - \tau(E^\perp) > 0.$$

\square

Let $\mathcal{P}(\mathcal{M})$ be the set of projections in \mathcal{M} . The following lemma says that above definition of s -numbers coincides with the definition of s -numbers given by Fack and Kosaki.

Lemma 2.6. For $0 \leq s < \infty$,

$$\mu_s(T) = \inf \{ \|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s \}.$$

Proof. By the polar decomposition and the definition of $\mu_s(T)$, we may assume that T is positive. Let \mathcal{A} be a separable diffuse abelian von Neumann subalgebra of \mathcal{M} containing T and let α be a $*$ -isomorphism from \mathcal{A} onto $L^\infty[0, \infty)$ such that $\tau = \int_0^\infty dx \cdot \alpha$. Let $f(x) =$

$\alpha(T)$ and let $f^*(x)$ be the nonincreasing rearrangement of $f(x)$. Then $\mu_s(T) = f^*(s)$. By the definition of f^* ,

$$m(\{f^* > \mu_s(T)\}) = \lim_{n \rightarrow \infty} m\left(\left\{f^* > \mu_s(T) + \frac{1}{n}\right\}\right) \leq s$$

and

$$m(\{f^* \geq \mu_s(T)\}) \geq \lim_{n \rightarrow \infty} m\left(\left\{f^* > \mu_s(T) - \frac{1}{n}\right\}\right) \geq s.$$

Since f^* and f are equi-measurable, $m(\{f > \mu_s(T)\}) \leq s$ and $m(\{f \geq \mu_s(T)\}) \geq s$. Therefore, there is a measurable set A of $[0, \infty)$, $\{f > \mu_s(T)\} \subset [0, \infty) \setminus A \subset \{f \geq \mu_s(T)\}$, such that $m([0, \infty) \setminus A) = s$ and $\|f(x)\chi_A(x)\|_\infty = \mu_s(T)$ and $\|f(x)\chi_B(x)\|_\infty \geq \mu_s(T)$ for every $B \subset [0, \infty) \setminus A$ such that $m(B) > 0$. Let $F = \alpha^{-1}(\chi_A)$. Then $\tau(F^\perp) = s$, $\|TF\| = \|\alpha^{-1}(f\chi_A)\|_\infty = \mu_s(T)$ and $\|TF'\| \geq \mu_s(T)$ for any nonzero subprojection F' of F^\perp . This proves that $\mu_s(T) \geq \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s\}$. Similarly, for any $\epsilon > 0$, there is a projection $F_\epsilon \in \mathcal{M}$ such that $\tau(F_\epsilon^\perp) = s + \epsilon$, $\|TF_\epsilon\| = \mu_{s+\epsilon}(T)$ and $\|TF'\| \geq \mu_{s+\epsilon}(T)$ for any nonzero subprojection F' of F_ϵ^\perp . Suppose $E \in \mathcal{M}$ is a projection such that $\tau(E^\perp) = s$. By Lemma 2.5, $\tau(E \wedge F_\epsilon^\perp) > 0$. Hence, $\|TE\| \geq \|T(E \wedge F_\epsilon^\perp)\| \geq \mu_{s+\epsilon}(T)$. This proves that $\inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s\} \geq \mu_{s+\epsilon}(T)$. Since $\mu_s(T)$ is right-continuous, $\mu_s(T) \leq \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s\}$. \square

Corollary 2.7. *Let $S, T \in \mathcal{M}$. Then $\mu_s(ST) \leq \|S\|\mu_s(T)$ for $s \in [0, \infty)$.*

We refer to [5, 6] for other interesting properties of s -numbers for operators in type II_∞ factors.

2.3 s -numbers of operators in semi-finite von Neumann algebras

An *embedding* of a semi-finite von Neumann algebra (\mathcal{M}, τ) into another semi-finite von Neumann algebra (\mathcal{M}_1, τ_1) means a $*$ -isomorphism α from \mathcal{M} to \mathcal{M}_1 such that $\tau = \tau_1 \cdot \alpha$. Every semi-finite von Neumann algebra can be embedded into a type II_∞ factor.

Definition 2.8. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and $T \in \mathcal{M}$. If α is an embedding of (\mathcal{M}, τ) into a type II_∞ factor (\mathcal{M}_1, τ_1) , then *the s -numbers of T* are defined as

$$\mu_s(T) = \mu_s(\alpha(T)), \quad 0 \leq s < \infty.$$

Similar to the proof of Lemma 2.3, we can see that $\mu_s(T)$ is well defined, i.e., does not depend on the choice of α and \mathcal{M}_1 .

Let $T \in (\mathcal{B}(\mathcal{H}), \text{Tr})$ be a finite rank operator, where \mathcal{H} is the separable infinite dimensional complex Hilbert space and Tr is the classical tracial weight on $\mathcal{B}(\mathcal{H})$. Then $|T|$ is

unitarily equivalent to a diagonal operator with diagonal elements $s_1(T) \geq s_2(T) \geq \cdots \geq 0$. In the classical operator theory [9], $s_1(T), s_2(T), \dots$ are also called s -numbers of T . It is easy to see that the relation between $\mu_s(T)$ and $s_1(T), s_2(T), \dots$ is the following

$$\mu_s(T) = s_1(T)\chi_{[0,1)}(s) + s_2(T)\chi_{[1,2)}(s) + \cdots. \quad (2)$$

Since no confusion will arise, we will use both s -numbers for a finite rank operator in $(\mathcal{B}(\mathcal{H}), \text{Tr})$. We refer to [1, 9] for properties of s -numbers for finite rank operators in $(\mathcal{B}(\mathcal{H}), \text{Tr})$.

We end this section by the following definition.

Definition 2.9. Two positive operators S, T in a semi-finite von Neumann algebra (\mathcal{M}, τ) are *equi-measurable* if $\mu_s(S) = \mu_s(T)$ for $0 \leq s < \infty$.

By 12 of Proposition 2.1 and Corollary 2.4, positive operators S and T in a semi-finite von Neumann algebra (\mathcal{M}, τ) are equi-measurable if and only if $\tau(S^n) = \tau(T^n)$ for all $n = 0, 1, 2, \dots$.

3 Semi-norms on $\mathcal{J}(\mathcal{M})$

In this section, (\mathcal{M}, τ) is a semi-finite von Neumann algebra with a faithful normal tracial weight τ . Recall that $\mathcal{J}(\mathcal{M})$ is the set of operators T in \mathcal{M} such that $T = ETE$ for some finite projection E . If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we simply write $\mathcal{J}(\mathcal{H})$ instead of $\mathcal{J}(\mathcal{B}(\mathcal{H}))$. Note that $\mathcal{J}(\mathcal{H})$ is the set of bounded linear operators T on \mathcal{H} such that both T and T^* are finite rank operators.

3.1 Gauge invariant semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 3.1. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra. A semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is *gauge invariant* if $\|T\| = \||T|\|$ for all $T \in \mathcal{J}(\mathcal{M})$. A semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is called *left unitarily invariant* if for all unitary operators U in \mathcal{M} and all T in $\mathcal{J}(\mathcal{M})$, $\|UT\| = \|T\|$.

Lemma 3.2. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and let $\|\cdot\|$ be a left unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$. If $T \in \mathcal{J}(\mathcal{M})$ and $A \in \mathcal{M}$, then $AT \in \mathcal{J}(\mathcal{M})$ and $\|AT\| \leq \|A\| \cdot \|T\|$.

Proof. It is easy to see that $AT \in \mathcal{J}(\mathcal{M})$. We need to prove that if $\|A\| < 1$, then $\|AT\| \leq \|T\|$. Since $\|A\| < 1$, there are unitary operators U_1, \dots, U_k such that $A = \frac{U_1 + \cdots + U_k}{k}$ (see [13, 20]). Since $\|\cdot\|$ is a left unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$,

$$\|AT\| = \left\| \frac{U_1 T + \cdots + U_k T}{k} \right\| \leq \frac{\|U_1 T\| + \cdots + \|U_k T\|}{k} \leq \|T\|.$$

□

Lemma 3.3. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and let $\|\cdot\|$ be a semi-norm on $\mathcal{J}(\mathcal{M})$. Then $\|\cdot\|$ is gauge invariant if and only if $\|\cdot\|$ is left unitarily invariant.*

Proof. Note that $|UT| = |T|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators U in \mathcal{M} . If $\|\cdot\|$ is gauge invariant then $\|\cdot\|$ is left unitarily invariant. Conversely, suppose $\|\cdot\|$ is left unitarily invariant. By Lemma 3.2, $\|T\| = \|V|T|\| \leq \| |T| \|$ and $\| |T| \| = \|V^*T\| \leq \|T\|$. Hence, $\|\cdot\|$ is gauge invariant. □

Corollary 3.4. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and let $\|\cdot\|$ be a gauge invariant semi-norm on $\mathcal{J}(\mathcal{M})$. If $T \in \mathcal{J}(\mathcal{M})$ and $0 \leq S \leq T$, then $S \in \mathcal{J}(\mathcal{M})$ and $\|S\| \leq \|T\|$.*

Proof. Since $0 \leq S \leq T$, there is an operator $A \in \mathcal{M}$ such that $S = AT$ and $\|A\| \leq 1$. By Lemma 3.2 and Lemma 3.3, $S \in \mathcal{J}(\mathcal{M})$ and $\|S\| = \|AT\| \leq \|A\| \cdot \|T\| \leq \|T\|$. □

3.2 Unitarily invariant semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 3.5. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra. A semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is *unitarily invariant* if $\|UTV\| = \|T\|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators $U, V \in \mathcal{M}$.

Proposition 3.6. *Let $\|\cdot\|$ be a semi-norm on $\mathcal{J}(\mathcal{M})$. Then the following statements are equivalent:*

1. $\|\cdot\|$ is unitarily invariant;
2. $\|\cdot\|$ is gauge invariant and unitarily conjugate invariant, i.e., $\|UTU^*\| = \|T\|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators $U \in \mathcal{M}$;
3. $\|\cdot\|$ is gauge invariant and $\|T\| = \|T^*\|$ for all $T \in \mathcal{J}(\mathcal{M})$;
4. for all operators $A, B \in \mathcal{M}$ and $T \in \mathcal{J}(\mathcal{M})$, $\|ATB\| \leq \|A\| \cdot \|T\| \cdot \|B\|$.

Proof. “1 \Rightarrow 4” is similar to the proof of Lemma 3.2.

“4 \Rightarrow 3”. Let $T = V|T|$. Then $T^* = |T|V^*$. By 4 and simple arguments, $\|T\| = \|T^*\|$.

“3 \Rightarrow 2”. By Lemma 3.3 and 3, $\|UTU^*\| = \|TU^*\| = \|UT^*\| = \|T^*\| = \|T\|$.

“2 \Rightarrow 1”. Suppose $\|\cdot\|$ is gauge invariant and unitarily conjugate invariant. Let $U, V \in \mathcal{M}$ be unitary operators and $T \in \mathcal{J}(\mathcal{M})$. By Lemma 3.3, $\|UTV\| = \|V^*VUTV\| = \|VUT\| = \|T\|$. □

Corollary 3.7. *Let $\|\cdot\|$ be a unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$ and let E, F be two equivalent projections in $\mathcal{J}(\mathcal{M})$. Then $\|E\| = \|F\|$.*

3.3 Symmetric gauge semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 3.8. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and let $\text{Aut}(\mathcal{M}, \tau)$ be the set of $*$ -automorphisms on \mathcal{M} preserving τ . A semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is called *symmetric* (with respect to τ) if

$$\|\theta(T)\| = \|T\|, \quad \forall T \in \mathcal{J}(\mathcal{M}), \theta \in \text{Aut}(\mathcal{M}, \tau);$$

a semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is called a *symmetric gauge semi-norm* if it is both symmetric and gauge invariant on $\mathcal{J}(\mathcal{M})$.

Example 3.9. The abelian von Neumann algebra \mathbb{C}^n is a finite von Neumann algebra with a classical tracial state $\tau((x_1, \dots, x_n)) = \frac{x_1 + \dots + x_n}{n}$. In this case, $\mathcal{J}(\mathbb{C}^n) = \mathbb{C}^n$. A norm $\|\cdot\|$ on \mathbb{C}^n is a symmetric gauge norm if and only if for every $(x_1, \dots, x_n) \in \mathbb{C}^n$,

1. $\|(x_1, \dots, x_n)\| = \|(|x_1|, \dots, |x_n|)\|$ and
2. $\|(x_1, \dots, x_n)\| = \|(x_{\pi(1)}, \dots, x_{\pi(n)})\|$ for every permutation π of $\{1, \dots, n\}$.

Example 3.10. The abelian von Neumann algebra $l^\infty(\mathbb{N})$ is a semi-finite von Neumann algebra with a classical tracial weight $\tau((x_1, x_2, \dots)) = x_1 + x_2 + \dots$. It is easy to see that $\mathcal{J}(l^\infty(\mathbb{N})) = c_{00}$ consists of (x_1, x_2, \dots) with $x_n = 0$ except for finitely many n . A norm $\|\cdot\|$ on $\mathcal{J}(l^\infty(\mathbb{N}))$ is a symmetric gauge norm if and only if for every $(x_1, x_2, \dots) \in c_{00}$,

1. $\|(x_1, x_2, \dots)\| = \|(|x_1|, |x_2|, \dots)\|$ and
2. $\|(x_1, x_2, \dots)\| = \|(x_{\pi(1)}, x_{\pi(2)}, \dots)\|$ for every permutation π of \mathbb{N} .

Example 3.11. The abelian von Neumann algebra $L^\infty[0, 1]$ is a finite von Neumann algebra with a classical tracial state $\tau = \int_0^1 dx$. In this case $\mathcal{J}(L^\infty[0, 1]) = L^\infty[0, 1]$. A norm $\|\cdot\|$ on $L^\infty[0, 1]$ is a symmetric gauge norm if and only if for every $f(x) \in L^\infty[0, 1]$,

1. $\|f(x)\| = \||f(x)|\|$ and
2. $\|f(x)\| = \|f(\phi(x))\|$ for every invertible measure preserving map ϕ of $[0, 1]$.

Example 3.12. The abelian von Neumann algebra $L^\infty[0, \infty)$ is a semi-finite von Neumann algebra with a classical tracial weight $\tau = \int_0^\infty dx$. A norm $\|\cdot\|$ on $\mathcal{J}(L^\infty[0, \infty))$ is a symmetric gauge norm if and only if for every $f(x) \in \mathcal{J}(L^\infty[0, \infty))$,

1. $\|f(x)\| = \||f(x)|\|$ and
2. $\|f(x)\| = \|f(\phi(x))\|$ for every invertible measure preserving map ϕ of $[0, \infty)$.

Lemma 3.13. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and let $\|\cdot\|$ be a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$. Then $\|\cdot\|$ is a unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$.

3.4 Symmetric gauge norms on (\mathcal{M}_E, τ_E)

In this paper we are interested symmetric gauge semi-norms on $\mathcal{J}(\mathcal{M})$, where (\mathcal{M}, τ) is one of the following semi-finite von Neumann algebras:

- $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \text{Tr}$ on $\mathcal{B}(\mathcal{H})$, where \mathcal{H} is the separable infinite dimensional complex Hilbert space;
- $\mathcal{M} = l^\infty(\mathbb{N})$ and $\tau((x_1, x_2, \dots)) = x_1 + x_2 + \dots$;
- \mathcal{M} is a type II_∞ factor and τ is a faithful normal tracial weight on \mathcal{M} ;
- $\mathcal{M} = L^\infty[0, \infty)$ and $\tau = \int_0^\infty dx$.

Note that in each case, $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically. Recall that $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically if $\theta(T) = T$ for all $\theta \in \text{Aut}(\mathcal{M}, \tau)$ implies $T = \lambda 1$. Let E, F be finite projections in \mathcal{M} such that $\tau(E) = \tau(F)$. If $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically, there is a $\theta \in \text{Aut}(\mathcal{M}, \tau)$ such that $\theta(E) = F$. Furthermore, if $\|\cdot\|$ is a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$, then $\|E\| = \|F\|$. In this case, a semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is called a *normalized symmetric gauge semi-norm* if $\|E\| = 1$ whenever $\tau(E) = 1$.

Let (\mathcal{M}, τ) be one of the above semi-finite von Neumann algebras. For every (non-zero) finite projection E in \mathcal{M} , let $\mathcal{M}_E = E\mathcal{M}E$ and $\tau_E(ETE) = \frac{\tau(ETE)}{\tau(E)}$. Then (\mathcal{M}_E, τ_E) is a finite von Neumann algebra satisfying the *weak Dixmier property* (see [4]), i.e., for every positive operator $T \in \mathcal{M}_E$, $\tau_E(T)E$ is in the operator norm closure of the convex hull of $\{S \in \mathcal{M}_E : S \text{ and } T \text{ are equi-measurable}\}$. So in the following sections we will always assume that (\mathcal{M}, τ) satisfies the following conditions:

- A.** (\mathcal{M}, τ) is a semi-finite von Neumann algebra such that $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically;
- B.** for every non-zero finite projection E in \mathcal{M} , (\mathcal{M}_E, τ_E) is a finite von Neumann algebra satisfying the weak Dixmier property.

With the above assumptions, it is easy to show that if E is a finite projection of \mathcal{M} , then $\text{Aut}(\mathcal{M}_E, \tau_E)$ acts on \mathcal{M}_E ergodically.

A *simple* operator in a semi-finite von Neumann algebra (\mathcal{M}, τ) is an operator $T = a_1 E_1 + \dots + a_n E_n$, where E_1, \dots, E_n are mutually orthogonal projections. The following lemma is Corollary 3.7 of [4].

Lemma 3.14. *Let \mathcal{N} be a finite von Neumann algebra with a faithful normal tracial state τ_N . Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two gauge invariant semi-norms on \mathcal{N} . Then $\|\cdot\|_1 = \|\cdot\|_2$ on \mathcal{N} if $\|T\|_1 = \|T\|_2$ for all positive simple operators $T \in \mathcal{N}$.*

Lemma 3.15. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra satisfying the conditions **A** and **B** and let $\|\cdot\|$ be a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$. If $E \in \mathcal{M}$ is a finite projection, then the restriction of $\|\cdot\|$ to (\mathcal{M}_E, τ_E) is also a symmetric gauge semi-norm on (\mathcal{M}_E, τ_E) .*

Proof. It is obvious that the restriction of $\|\cdot\|$ to (\mathcal{M}_E, τ_E) is also a gauge semi-norm on (\mathcal{M}_E, τ_E) . Let $\theta \in \text{Aut}(\mathcal{M}_E, \tau_E)$. Define $\|S\|_2 = \|\theta(S)\|$ for $S \in \mathcal{M}_E$. We need to prove $\|\cdot\| = \|\cdot\|_2$ on \mathcal{M}_E . Let $T = a_1 E_1 + \cdots + a_n E_n$ be a simple positive operator in \mathcal{M}_E , where $E_1 + \cdots + E_n = E$. Then $\theta(T) = a_1 \theta(E_1) + \cdots + a_n \theta(E_n)$. Since $\theta \in \text{Aut}(\mathcal{M}_E, \tau_E)$, $\tau(E_k) = \tau(\theta(E_k))$ for $1 \leq k \leq n$. By the assumption of the lemma, $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically. Therefore, there is a $\theta' \in \text{Aut}(\mathcal{M}, \tau)$ such that $\theta'(E_k) = \theta(E_k)$ for $1 \leq k \leq n$. Hence, $\theta'(T) = \theta(T)$. Since $\|\cdot\|$ is a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$, $\|T\| = \|\theta'(T)\| = \|\theta(T)\| = \|T\|_2$. By Lemma 3.14, $\|\cdot\| = \|\cdot\|_2$ on (\mathcal{M}_E, τ_E) . This implies that the restriction of $\|\cdot\|$ to (\mathcal{M}_E, τ_E) is also a symmetric gauge semi-norm on (\mathcal{M}_E, τ_E) . \square

The following lemma is Theorem 3.27 of [4].

Lemma 3.16. *Let \mathcal{N} be a finite von Neumann algebra with a faithful normal tracial state τ_N . Then \mathcal{N} satisfies the weak Dixmier property if and only if \mathcal{N} satisfies one of the following conditions:*

1. \mathcal{N} is finite dimensional (hence atomic) and for every two non-zero minimal projections $E, F \in \mathcal{N}$, $\tau(E) = \tau(F)$, or equivalently, (\mathcal{N}, τ_N) can be identified as a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ that contains all diagonal matrices;
2. \mathcal{N} is diffuse.

Corollary 3.17. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra satisfying the conditions **A** and **B** and let $\|\cdot\|$ be a normalized symmetric gauge norm on \mathcal{M} . If F is a finite projection in \mathcal{M} such that $\tau(F) \geq 1$, then $\|F\| \geq 1$.*

Proof. Let $E_1 \in \mathcal{M}$ be a finite projection such that $\tau(E_1) = 1$ and $\|E_1\| = 1$. There exists a finite projection $E \in \mathcal{M}$ such that $E_1, F \leq E$. By Lemma 3.15, the restriction of $\|\cdot\|$ to (\mathcal{M}_E, τ_E) is also a symmetric gauge semi-norm on (\mathcal{M}_E, τ_E) . Since \mathcal{M}_E satisfies the weak Dixmier property, there is a projection $F_1 \in \mathcal{M}_E$ such that $F_1 \leq F$ and $\tau(F_1) = 1$ by Lemma 3.16. Since $\text{Aut}(\mathcal{M}_E, \tau_E)$ acts on \mathcal{M}_E ergodically, $\|F_1\| = \|E_1\| = 1$. By Corollary 3.4, $\|F\| \geq \|F_1\| = 1$. \square

The following lemma is Corollary 3.36 of [4]

Lemma 3.18. *Let \mathcal{N} be a finite von Neumann algebra with a faithful normal tracial state τ_N . Suppose \mathcal{N} satisfies the weak Dixmier property and $\text{Aut}(\mathcal{N}, \tau)$ acts on \mathcal{N} ergodically. If $\|\cdot\|$ is a symmetric gauge semi-norm on \mathcal{N} , then for every $T \in \mathcal{N}$,*

$$\|T\|_1 \cdot \|1\| \leq \|T\| \leq \|T\| \cdot \|1\|,$$

where $\|T\|_1 = \tau_N(|T|)$.

Corollary 3.19. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra satisfying the conditions **A** and **B**. If $\|\cdot\|$ is a normalized symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$, then $\|\cdot\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$.*

Proof. Let $T \in \mathcal{J}(\mathcal{M})$. Then there is a finite projection E in \mathcal{M} such that $T = ETE \in (\mathcal{M}_E, \tau_E)$. We may assume that $\tau(E) \geq 1$. By Lemma 3.15, the restriction of $\|\cdot\|$ to (\mathcal{M}_E, τ_E) is also a symmetric gauge semi-norm. If $T \neq 0$, then by Lemma 3.18 and Corollary 3.17, $\|T\| \geq \tau_E(|T|) \cdot \|E\| > 0$. So $\|\cdot\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. \square

The following lemma is Corollary 4.4 of [4].

Lemma 3.20. *Let \mathcal{N} be a finite von Neumann algebra with a faithful normal tracial state τ_N , and let $\|\cdot\|_1, \|\cdot\|_2$ be two symmetric gauge norms on \mathcal{N} . Suppose \mathcal{N} satisfies the weak Dixmier property and $\text{Aut}(\mathcal{N}, \tau)$ acts on \mathcal{N} ergodically. Then $\|\cdot\|_1 = \|\cdot\|_2$ on \mathcal{N} if $\|T\|_1 = \|T\|_2$ for every operator $T = a_1 E_1 + \cdots + a_n E_n$ in \mathcal{N} such that $a_1, \dots, a_n \geq 0$ and $\tau_N(E_1) = \cdots = \tau_N(E_n) = \frac{1}{n}$, $n = 1, 2, \dots$.*

Lemma 3.21. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra satisfying the conditions **A** and **B**. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. Then $\|\cdot\|_1 = \|\cdot\|_2$ on $\mathcal{J}(\mathcal{M})$ if $\|T\|_1 = \|T\|_2$ for every simple positive operator T in $\mathcal{J}(\mathcal{M})$ such that $T = a_1 E_1 + \cdots + a_n E_n$ and $\tau(E_1) = \cdots = \tau(E_n)$.*

Proof. Suppose $\|T\|_1 = \|T\|_2$ for every simple operator T in $\mathcal{J}(\mathcal{M})$. Let $S \in \mathcal{J}(\mathcal{M})$. Then there is a finite projection E in \mathcal{M} such that $S = ESE \in \mathcal{M}_E$. By Lemma 3.15, the restrictions of $\|\cdot\|_1$ and $\|\cdot\|_2$ to (\mathcal{M}_E, τ_E) are two symmetric gauge norms. Since $\|T\|_1 = \|T\|_2$ for every simple operator T in \mathcal{M}_E such that $T = a_1 E_1 + \cdots + a_n E_n$ and $\tau(E_1) = \cdots = \tau(E_n)$, $\|\cdot\|_1 = \|\cdot\|_2$ by Lemma 3.20. \square

Proposition 3.22. *Let (\mathcal{M}, τ) be a semi-finite factor and let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$. Then the following conditions are equivalent:*

1. $\|\cdot\|$ is a symmetric gauge norm;
2. $\|\cdot\|$ is a unitarily invariant norm.

Proof. “1 \Rightarrow 2” is obvious. We only prove “2 \Rightarrow 1”. We need to prove that for every positive operator $T \in \mathcal{J}(\mathcal{M})$ and $\theta \in \text{Aut}(\mathcal{M}, \tau)$, $\| \theta(T) \| = \| T \|$. Let $S = \theta(T)$. Then $S \in \mathcal{J}(\mathcal{M})$. Therefor, there is a finite projection E in \mathcal{M} such that $S, T \in \mathcal{M}_E$. By the spectral decomposition theorem, there is a sequence of simple positive operators $T_n \in \mathcal{M}_E$ such that $S_n = \theta(T_n) \in \mathcal{M}_E$ and $\lim_{n \rightarrow \infty} \|T_n - T\| = \lim_{n \rightarrow \infty} \|S_n - S\| = 0$. By Lemma 3.18, $\|T - T_n\| \leq \|T - T_n\| \cdot \|E\|$ and $\|S - S_n\| \leq \|S - S_n\| \cdot \|E\|$. Hence, $\lim_{n \rightarrow \infty} \|T - T_n\| = \lim_{n \rightarrow \infty} \|S - S_n\| = 0$. We need only prove $\|T_n\| = \|S_n\|$ for all $n = 1, 2, \dots$. Suppose $T_n = a_1 E_1 + \dots + a_m E_m$. Then $S_n = \theta(T_n) = a_1 F_1 + \dots + a_m F_m$, where $\theta(E_k) = F_k$ for $1 \leq k \leq m$. Since $\theta \in \text{Aut}(\mathcal{M}, \tau)$, $\tau(E_k) = \tau(F_k)$ for $1 \leq k \leq m$. Since \mathcal{M} is a factor, there is a unitary operator $U \in \mathcal{M}$ such that $E_k = U F_k U^*$ for $1 \leq k \leq m$. Therefore, $S_n = U T_n U^*$ and $\|T_n\| = \|S_n\|$. \square

3.5 Semi-norms associated to von Neumann algebras

Definition 3.23. Let \mathcal{M} be a von Neumann algebra (not necessarily semi-finite). A (generalized) semi-norm associated to \mathcal{M} is a map $\| \cdot \|$ from \mathcal{M} to $[0, \infty]$ satisfying the following properties:

1. $\| \lambda T \| = |\lambda| \cdot \| T \|$,
2. $\| S + T \| \leq \| S \| + \| T \|$

for all $S, T \in \mathcal{M}$ and $\lambda \in \mathbb{C}$. To make the definition nontrivial, we always make the following assumption: $0 < \| T \| < \infty$ for some non-zero element $T \in \mathcal{M}$.

Let $\mathcal{I} = \{ T \in \mathcal{M} : \| T \| < \infty \}$. Then \mathcal{I} is called *the domain* of the semi-norm $\| \cdot \|$.

Definition 3.24. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra. A semi-norm $\| \cdot \|$ associated to \mathcal{M} is called *gauge invariant* if for all $T \in \mathcal{M}$, $\| T \| = \| |T| \|$; a semi-norm $\| \cdot \|$ associated to \mathcal{M} is *unitarily invariant* if $\| U T V \| = \| T \|$ for all $T \in \mathcal{M}$ and unitary operators $U, V \in \mathcal{M}$; a semi-norm $\| \cdot \|$ associated to a semi-finite von Neumann algebra (\mathcal{M}, τ) is called *symmetric* if

$$\| \theta(T) \| = \| T \|, \quad \forall T \in \mathcal{M}, \theta \in \text{Aut}(\mathcal{M}, \tau);$$

a semi-norm $\| \cdot \|$ associated to (\mathcal{M}, τ) is called a *symmetric gauge semi-norm* if it is both symmetric and gauge invariant.

Similar to the proof of Proposition 3.6, we can prove the following proposition.

Proposition 3.25. Let $\| \cdot \|$ be a semi-norm associated to \mathcal{M} . Then the following statements are equivalent:

1. $\|\cdot\|$ is unitarily invariant;
2. $\|\cdot\|$ is gauge invariant and unitarily conjugate invariant, i.e., $\|UTU^*\| = \|T\|$ for all $T \in \mathcal{M}$ and unitary operators $U \in \mathcal{M}$;
3. $\|\cdot\|$ is gauge invariant and $\|T\| = \|T^*\|$ for all $T \in \mathcal{M}$;
4. for all operators $T, A, B \in \mathcal{M}$, $\|ATB\| \leq \|A\| \cdot \|T\| \cdot \|B\|$.

Corollary 3.26. *Let $\|\cdot\|$ be a semi-norm associated to \mathcal{M} . If $S, T \in \mathcal{M}$ and $0 \leq S \leq T$, then $\|S\| \leq \|T\|$.*

Corollary 3.27. *Let $\|\cdot\|$ be a unitarily invariant semi-norm associated to \mathcal{M} and E, F be two equivalent projections in \mathcal{M} . Then $\|E\| = \|F\|$.*

Lemma 3.28. *Let $\|\cdot\|$ be a unitarily invariant semi-norm associated to \mathcal{M} and let $T \in \mathcal{M}$ be a nonzero element such that $\|T\| < \infty$. Then there is a nonzero projection E in \mathcal{M} such that $\|E\| < \infty$.*

Proof. Since $\|\cdot\|$ is unitarily invariant, we may assume $T > 0$. By the spectral decomposition theorem, there exist a $\lambda > 0$ and a nonzero projection E in \mathcal{M} such that $T \geq \lambda E$. By Corollary 3.26, $\|E\| < \infty$. \square

The following theorem shows that, up to a scale $a > 0$, the operator norm $\|\cdot\|$ is the unique unitarily invariant semi-norm associated to a type III factor.

Theorem 3.29. *Let \mathcal{M} be a type III factor and let $\|\cdot\|$ be a unitarily invariant semi-norm associated to \mathcal{M} . Then there exists $a > 0$ such that $\|\cdot\| = a\|\cdot\|$, i.e., $\|T\| = a\|T\|$ for all $T \in \mathcal{M}$.*

Proof. By Lemma 3.28, there is a nonzero projection E in \mathcal{M} such that $\|E\| < \infty$. If $\|E\| = 0$, then $\|1\| = 0$ by Corollary 3.27. By Proposition 3.25, for every T in \mathcal{M} , $\|T\| \leq \|T\| \cdot \|1\| = 0$. In our definition of semi-norm, we assume that $\|T\| > 0$ for some $T \in \mathcal{M}$. Hence $\|E\| \neq 0$ for some projection E in \mathcal{M} . We may assume that $\|E\| = 1$. By Corollary 3.27, $\|F\| = 1$ for every non-zero projection in \mathcal{M} . In particular, $\|1\| = 1$. By Proposition 3.25, for every T in \mathcal{M} , $\|T\| \leq \|T\| \cdot \|1\| = \|T\|$. On the other hand, let $T \in \mathcal{M}$ be a positive operator and $\epsilon > 0$. By the spectral decomposition theorem, there is a nonzero projection F in \mathcal{M} such that $T \geq (\|T\| - \epsilon)F$. By Corollary 3.26, $\|T\| \geq (\|T\| - \epsilon) \cdot \|F\| = \|T\| - \epsilon$. Hence $\|T\| \geq \|T\| - \epsilon$. This proves that $\|T\| = \|T\|$ for every positive operator T in \mathcal{M} and therefore for every operator T in \mathcal{M} . \square

We end this section with the following lemma.

Lemma 3.30. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra such that $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically. If $\|\cdot\|$ is a normalized symmetric gauge semi-norm associated to \mathcal{M} with domain \mathcal{I} , then $\mathcal{I} \supseteq \mathcal{J}(\mathcal{M})$ and $\|\cdot\|$ is a normalized symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$.*

Proof. Let E be a finite projection in \mathcal{M} such that $\tau(E) = 1$. Then $\|E\| = 1$. Suppose that F is a finite projection in \mathcal{M} such that $n \leq \tau(F) < n+1$. Note that $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically. By induction, there are mutually orthogonal finite projections E_1, E_2, \dots, E_{n+1} in \mathcal{M} , $\tau(E_1) = \dots = \tau(E_{n+1}) = 1$, such that $E_1 + \dots + E_n \leq F \leq E_1 + \dots + E_{n+1}$. By Corollary 3.26, $\|F\| \leq \|E_1 + \dots + E_{n+1}\| \leq n+1$. So every finite projection is in \mathcal{I} . Hence $\mathcal{I} \supseteq \mathcal{J}(\mathcal{M})$. \square

4 Ky Fan norms associated to semi-finite von Neumann algebras

Let (\mathcal{M}, τ) be a semi-finite von Neumann subalgebra of a type II_∞ factor (\mathcal{M}_1, τ_1) and let $0 \leq t \leq \infty$. For $T \in \mathcal{M}$, define $\|T\|_{(t)}$, the Ky Fan t -th norm of T , by

$$\|T\|_{(t)} = \begin{cases} \|T\|, & t = 0; \\ \frac{1}{t} \int_0^t \mu_s(T) ds, & 0 < t \leq 1; \\ \int_0^t \mu_s(T) ds, & 1 < t \leq \infty. \end{cases}$$

Let $\mathcal{U}(\mathcal{M})$ be the set of unitary operators in \mathcal{M} and $\mathcal{P}(\mathcal{M})$ be the set of projections in \mathcal{M} .

Lemma 4.1. *For $0 < t \leq 1$, $t\|T\|_{(t)} = \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\}$.*

Proof. We may assume that T is a positive operator. Let \mathcal{A} be a separable diffuse abelian von Neumann subalgebra of \mathcal{M}_1 containing T and let α be a $*$ -isomorphism from (\mathcal{A}, τ_1) onto $(L^\infty[0, \infty), \int_0^\infty dx)$ such that $\tau_1 = \int_0^\infty dx \cdot \alpha$. Let $f(x) = \alpha(T)$ and let $f^*(x)$ be the non-increasing rearrangement of $f(x)$. Then $\mu_s(T) = f^*(s)$. By the definition of f^* (see equation (1) in section 2.1),

$$m(\{f^* > f^*(t)\}) = \lim_{n \rightarrow \infty} m\left(\left\{f^* > f^*(t) + \frac{1}{n}\right\}\right) \leq t$$

and

$$m(\{f^* \geq f^*(t)\}) \geq \lim_{n \rightarrow \infty} m\left(\left\{f^* > f^*(t) - \frac{1}{n}\right\}\right) \geq t.$$

Since f^* and f are equi-measurable, $m(\{f > f^*(t)\}) \leq t$ and $m(\{f \geq f^*(t)\}) \geq t$. Therefore, there is a measurable subset A of $[0, \infty)$, $\{f > f^*(t)\} \subset A \subset \{f \geq f^*(t)\}$, such that $m(A) = t$. Since $f(x)$ and $f^*(x)$ are equi-measurable, $\int_A f(s)ds = \int_0^t f^*(s)ds$. Let $E' = \alpha^{-1}(\chi_A)$. Then $\tau_1(E') = t$ and $\tau_1(TE') = \int_A f(s)ds = \int_0^t f^*(s)ds = t\|T\|_{(t)}$. Hence, $t\|T\|_{(t)} \leq \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\}$.

We need to prove that if E is a projection in \mathcal{M}_1 , $\tau_1(E) = t$, and $U \in \mathcal{U}(\mathcal{M}_1)$, then $t\|T\|_{(t)} \geq |\tau_1(UTE)|$. By the Schwartz inequality, $|\tau_1(UTE)| = |\tau_1(EUT^{1/2}T^{1/2}E)| \leq \tau_1(U^*EUT)^{1/2}\tau_1(ET)^{1/2}$. By Corollary 2.4, $\tau_1(ET) = \int_0^1 \mu_s(ET)ds$. By Corollary 2.7, $\mu_s(ET) \leq \min\{\mu_s(T), \mu_s(E)\|T\|\}$. Note that $\mu_s(E) = 0$ for $s \geq \tau_1(E) = t$. Hence, $\tau_1(ET) \leq \int_0^t \mu_s(T)ds = t\|T\|_t$. Similarly, $\tau_1(U^*EUT) \leq t\|T\|_t$. So $|\tau_1(UTE)| \leq t\|T\|_t$. This proves that $t\|T\|_{(t)} \geq \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\}$. \square

Similarly, we can prove the following lemma.

Lemma 4.2. *For $1 \leq t \leq \infty$, $\|T\|_{(t)} = \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\}$.*

Theorem 4.3. *For $0 \leq t \leq \infty$, $\|\cdot\|_{(t)}$ is a normalized symmetric gauge norm associated to (\mathcal{M}, τ) .*

Proof. By the definition of s -number, $\mu_s(T) = \mu_s(\theta(T))$ for $T \in \mathcal{M}$ and $\theta \in \text{Aut}(\mathcal{M}, \tau)$. To prove $\|\cdot\|_{(t)}$ is a normalized symmetric gauge norm associated to (\mathcal{M}, τ) , we need only prove the triangle inequality since other parts are obvious. Let $S, T \in \mathcal{M}$. If $0 < t \leq 1$, by Lemma 4.1, $t\|S + T\|_{(t)} = \sup\{|\tau_1(U(S + T)E)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\} \leq \sup\{|\tau_1(USE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\} + \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\} = t\|S\|_{(t)} + t\|T\|_{(t)}$. The proof of the case $t > 1$ is similar. \square

The following corollary plays a key role in section 10 (see the proof of Lemma 10.6).

Corollary 4.4. *Let $T \in \mathcal{M}$ and $\delta > 0$. If $\|T\|_{(1)} < \delta$, then $\tau(\chi_{(\delta, \infty)}(|T|)) \leq \frac{\|T\|_{(1)}}{\delta}$.*

Proof. We may assume that \mathcal{M} is a type II_∞ factor and $T \geq 0$. By the proof of Lemma 4.1,

$$\|T\|_{(1)} = \sup\{|\tau(UTE)| : U \in \mathcal{U}(\mathcal{M}), E \in \mathcal{P}(\mathcal{M}), \tau(E) \leq 1\}.$$

If $\tau(\chi_{(\delta, \infty)}(T)) > 1$, then there is a sub-projection E of $\chi_{(\delta, \infty)}(T)$ such that $\tau(E) = 1$. Then $TE \geq \delta E$. Hence, $\|T\|_{(1)} \geq \tau(TE) \geq \tau(\delta E) = \delta$. This contradicts the assumption that $\|T\|_{(1)} < \delta$. Therefore, $\tau(\chi_{(\delta, \infty)}(T)) \leq 1$. So $\|T\|_{(1)} \geq \tau(T\chi_{(\delta, \infty)}(T)) \geq \tau(\delta\chi_{(\delta, \infty)}(T)) \geq \delta\tau(\chi_{(\delta, \infty)}(T))$. This implies the corollary. \square

Proposition 4.5. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and $T \in (\mathcal{M}, \tau)$. Then $\|T\|_{(t)}$ is a non-increasing continuous function on $[0, 1]$ and a non-decreasing continuous function on $[1, \infty]$.*

Proof. Let $0 < t_1 < t_2 \leq 1$. $\|T\|_{(t_1)} - \|T\|_{(t_2)} = \frac{1}{t_1} \int_0^{t_1} \mu_s(T) ds - \frac{1}{t_2} \int_0^{t_2} \mu_s(T) ds = \frac{\frac{1}{t_1} \int_0^{t_1} \mu_s(T) ds - \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mu_s(T) ds}{t_2(t_2 - t_1)} \leq 0$. Since $\mu_s(T)$ is right-continuous, $\|T\|_{(t)}$ is a non-increasing continuous function on $[0, 1]$. Since $\mu_s(T) \geq 0$ for $s \in [0, \infty)$, $\|T\|_{(t)}$ is a non-decreasing continuous function on $[1, \infty]$. \square

Proposition 4.6. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra satisfying the conditions **A** and **B** in section 3.4, and let $\|\cdot\|$ be a normalized symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Then for every $T \in \mathcal{J}(\mathcal{M})$,*

$$\|T\|_{(1)} \leq \|T\|.$$

Proof. We can assume that T is a positive operator in $\mathcal{J}(\mathcal{M})$. Then there is a finite projection F in \mathcal{M} such that $T = FTF \in \mathcal{M}_F$. We can assume that $\tau(F) = k$ is a positive integer. By the assumption of the proposition, (\mathcal{M}_F, τ_F) satisfies the weak Dixmier property. By Lemma 3.16, either (\mathcal{M}_F, τ_F) is a diffuse von Neumann algebra or (\mathcal{M}_F, τ_F) is $*$ -isomorphic to a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ that contains all diagonal matrices. In either case, there is a projection E in \mathcal{M} , $E \leq F$, such that $\tau(E) = 1$ and $\|T\|_{(1)} = \|ETE\|_{(1)}$. By Corollary 3.13 and Proposition 3.6, $\|ETE\| \leq \|T\|$. By Lemma 3.18, $\|ETE\|_{(1)} \leq \|ETE\| \leq \|T\|$. \square

Example 4.7. The Ky Fan n -th norm of a compact operator $T \in (\mathcal{B}(\mathcal{H}), \text{Tr})$ is

$$\|T\|_{(n)} = s_1(T) + \cdots + s_n(T)$$

and

$$\|T\|_{(\infty)} = s_1(T) + s_2(T) + \cdots.$$

Corollary 4.8. *Let $\|\cdot\|$ be a normalized unitarily invariant norm on $\mathcal{B}(\mathcal{H})$. Then for every $T \in \mathcal{J}(\mathcal{H})$,*

$$s_1(T) \leq \|T\| \leq s_1(T) + s_2(T) + \cdots.$$

Proof. By Proposition 4.6, $s_1(T) = \|T\|_{(1)} \leq \|T\|$. On the other hand, we may assume that T is a positive operator in $\mathcal{J}(\mathcal{H})$. Then T is unitarily equivalent to a diagonal operator $s_1(T)E_1 + \cdots + s_n(T)E_n$. Hence, $\|T\| = \|s_1(T)E_1 + \cdots + s_n(T)E_n\| \leq s_1(T) + \cdots + s_n(T)$. \square

5 Dual norms of symmetric gauge norms on $\mathcal{J}(\mathcal{M})$

Throughout this section, we assume that (\mathcal{M}, τ) is a semi-finite von Neumann algebra satisfying the conditions **A** and **B** in section 3.4. Recall that $\mathcal{J}(\mathcal{M})$ is the subset of \mathcal{M} consisting of operators T in \mathcal{M} such that $T = ETE$ for some finite projection $E \in \mathcal{M}$. Note that for two operators S, T in $\mathcal{J}(\mathcal{M})$, there is a finite projection F in \mathcal{M} such that $S, T \in \mathcal{M}_F = F\mathcal{M}F$.

5.1 Dual norms

Let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$. For $T \in \mathcal{J}(\mathcal{M})$, define

$$\|T\|_{\mathcal{M}, \tau}^{\#} = \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}), \|X\| \leq 1\}.$$

When no confusion arises, we simply write $\|\cdot\|^{\#}$ or $\|\cdot\|_{\mathcal{M}}^{\#}$ instead of $\|\cdot\|_{\mathcal{M}, \tau}^{\#}$.

Lemma 5.1. $\|\cdot\|^{\#}$ is a norm on $\mathcal{J}(\mathcal{M})$.

Proof. Note that if $T \in \mathcal{J}(\mathcal{M})$ is not 0, then $\|T\|^{\#} \geq \frac{\tau(TT^*)}{\|T^*\|} > 0$. It is easy to check that $\|\cdot\|^{\#}$ satisfies other conditions for a norm. \square

Definition 5.2. $\|\cdot\|^{\#}$ is called the *dual norm* of $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ with respect to τ .

The following lemma follows simply from the definition of dual norm.

Lemma 5.3. Let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|^{\#}$ be the dual norm on $\mathcal{J}(\mathcal{M})$. Then for $S, T \in \mathcal{J}(\mathcal{M})$, $|\tau(ST)| \leq \|S\| \cdot \|T\|^{\#}$.

For $T \in \mathcal{M}$, define $\|T\|_1 = \tau(|T|)$. Then $\|T\|_1 = \|T\|_{(\infty)}$. The following corollary is the Hölder's inequality for operators in $\mathcal{J}(\mathcal{M})$.

Corollary 5.4. Let $\|\cdot\|$ be a gauge invariant norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|^{\#}$ be the dual norm. Then for $S, T \in \mathcal{J}(\mathcal{M})$, $\|ST\|_1 \leq \|S\| \cdot \|T\|^{\#}$.

Proof. By Lemma 4.2, $\|ST\|_1 = \|ST\|_{(\infty)} = \sup\{|\tau(UST)| : U \in \mathcal{U}(\mathcal{M})\}$. By Lemma 5.3 and Lemma 3.3, $|\tau(UST)| \leq \|US\| \cdot \|T\|^{\#} = \|S\| \cdot \|T\|^{\#}$. \square

Let E be a (non-zero) finite projection in \mathcal{M} . Recall that $\mathcal{M}_E = E\mathcal{M}E$ is a finite von Neumann algebra with a faithful normal tracial state τ_E such that $\tau_E(T) = \frac{\tau(T)}{\tau(E)}$ for $T \in \mathcal{M}_E$. If $\|\cdot\|$ is a norm on \mathcal{M}_E , the dual norm of $T \in \mathcal{M}_E$ with respect to τ_E is defined by

$$\|T\|_{\mathcal{M}_E, \tau_E}^{\#} = \sup\{|\tau_E(TX)| : X \in \mathcal{M}_E, \|X\| \leq 1\}.$$

Lemma 5.5. *Suppose $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. Let E be a non-zero finite projection in \mathcal{M} and $T \in \mathcal{M}_E$. Then*

$$\|T\|_{\mathcal{M},\tau}^{\#} = \tau(E) \cdot \|T\|_{\mathcal{M}_E,\tau_E}^{\#}.$$

Proof. Since $T = ETE$, for every $X \in \mathcal{J}(\mathcal{M})$, $\tau(TX) = \tau(ETEX) = \tau(ETEEEXE) = \tau(E) \cdot \tau_E(ETEEEXE)$. If $\|X\| \leq 1$, then $\|EXE\| \leq \|X\|$ by Proposition 3.6. This implies that

$$\begin{aligned} \|T\|_{\mathcal{M},\tau}^{\#} &= \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}), \|X\| \leq 1\} = \sup\{|\tau(TX)| : X \in \mathcal{M}_E, \|X\| \leq 1\} \\ &= \tau(E) \cdot \sup\{|\tau_E(TX)| : X \in \mathcal{M}_E, \|X\| \leq 1\} = \tau(E) \cdot \|T\|_{\mathcal{M}_E,\tau_E}^{\#}. \end{aligned}$$

□

The next lemma follows from Proposition 6.5, Proposition 6.6 and Theorem 6.10 of [4].

Lemma 5.6. *Let \mathcal{N} be a finite von Neumann algebra with a faithful normal tracial state $\tau_{\mathcal{N}}$. We have the following:*

1. *if $\|\cdot\|$ is a unitarily invariant norm on \mathcal{N} , then $\|\cdot\|_{\mathcal{N},\tau_{\mathcal{N}}}^{\#}$ is also a unitarily invariant norm on \mathcal{N} ;*
2. *if $\|\cdot\|$ is a symmetric gauge norm on \mathcal{N} , then $\|\cdot\|_{\mathcal{N},\tau_{\mathcal{N}}}^{\#}$ is also a symmetric gauge norm on \mathcal{N} . Furthermore, if $\|1\| = 1$, then $\|1\|_{\mathcal{N},\tau_{\mathcal{N}}}^{\#} = 1$.*

Combining Lemma 5.5 and Lemma 5.6, we have the following proposition.

Proposition 5.7. *Let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$. We have the following:*

1. *if $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$, then $\|\cdot\|^{\#}$ is also a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$;*
2. *if $\|\cdot\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then $\|\cdot\|^{\#}$ is also a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Furthermore, if $\|\cdot\|$ is a normalized norm, i.e., $\|E\| = 1$ if $\tau(E) = 1$, then $\|\cdot\|^{\#}$ is also a normalized norm.*

The next Lemma follows from Lemma 6.9 of [4].

Lemma 5.8. *Let \mathcal{N} be a finite von Neumann algebra with a faithful normal tracial state $\tau_{\mathcal{N}}$. Suppose \mathcal{N} satisfies the weak Dixmier property and $\|\cdot\|$ is a symmetric gauge norm on \mathcal{N} . If $T = a_1E_1 + \cdots + a_nE_n$ is a positive simple operator in \mathcal{N} , then*

$$\|T\|_{\mathcal{N},\tau_{\mathcal{N}}}^{\#} = \sup \left\{ \sum_{k=1}^n a_k b_k \tau(E_k) : X = b_1E_1 + \cdots + b_nE_n \geq 0 \text{ and } \|X\| \leq 1 \right\}.$$

Lemma 5.9. *Let $\|\cdot\|$ be a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. If $T = a_1 E_1 + \cdots + a_n E_n$ is a positive simple operator in $\mathcal{J}(\mathcal{M})$, then*

$$\|T\|^\# = \sup \left\{ \sum_{k=1}^n a_k b_k \tau(E_k) : S = b_1 E_1 + \cdots + b_n E_n \geq 0, \|S\| \leq 1 \right\}.$$

Proof. Let $E = E_1 + \cdots + E_n$. By Lemma 5.5 and Lemma 5.8,

$$\begin{aligned} \|T\|^\# &= \tau(E) \cdot \|T\|_{\mathcal{M}_E, \tau_E}^\# \\ &= \tau(E) \sup \left\{ \sum_{k=1}^n a_k b_k \tau_E(E_k) : S = b_1 E_1 + \cdots + b_n E_n \geq 0, \|S\| \leq 1 \right\} \\ &= \sup \left\{ \sum_{k=1}^n a_k b_k \tau(E_k) : S = b_1 E_1 + \cdots + b_n E_n \geq 0, \|S\| \leq 1 \right\}. \end{aligned}$$

□

5.2 Dual norms of Ky Fan norms

The following lemma is Lemma 6.11 of [4].

Lemma 5.10. *For $T \in (M_n(\mathbb{C}), \tau_n)$,*

$$\|T\|_{(\frac{k}{n}), M_n(\mathbb{C}), \tau_n}^\# = \max \left\{ \frac{k}{n} \|T\|, \|T\|_{1, \tau_n} \right\},$$

where $\|T\|_{(\frac{k}{n}), M_n(\mathbb{C}), \tau_n} = \frac{s_1(T) + \cdots + s_k(T)}{k}$ and $\|T\|_{1, \tau_n} = \tau_n(|T|) = \frac{s_1(T) + \cdots + s_n(T)}{n}$.

Theorem 5.11. *For $T \in \mathcal{J}(\mathcal{H})$ and $k = 1, 2, \dots, \infty$,*

$$\|T\|_{(k)}^\# = \max \left\{ \|T\|, \frac{1}{k} \|T\|_1 \right\},$$

where $\|T\|_{(k)} = s_1(T) + \cdots + s_k(T)$, $\|T\|_1 = \text{Tr}(|T|) = s_1(T) + s_2(T) + \cdots$ and $\frac{1}{\infty} = 0$.

Proof. For $T \in \mathcal{J}(\mathcal{H})$, there is a finite rank projection E such that $T = ETE \in \mathcal{B}(\mathcal{H})_E$. Let $\text{Tr}(E) = n$. Then $\mathcal{B}(\mathcal{H})_E \cong M_n(\mathbb{C})$. First assume $k < \infty$. We may assume that $n \geq k$. Then $\|T\|_{(k)} = k \|T\|_{(\frac{k}{n}), \tau_n}$. By Lemma 5.5 and Lemma 5.10,

$$\|T\|_{(k)}^\# = \text{Tr}(E) \cdot \left(k \|T\|_{(\frac{k}{n}), M_n(\mathbb{C}), \tau_n}^\# \right) = \frac{n}{k} \max \left\{ \frac{k}{n} \|T\|, \|T\|_{1, \tau_n} \right\} = \max \left\{ \|T\|, \frac{1}{k} \|T\|_1 \right\}.$$

If $k = \infty$, then $\|T\|_{(\infty)}^\# = \|T\|_{(n)}^\#$ by Lemma 5.9. Since $\frac{1}{n} \|T\|_1 \leq \|T\|$, $\|T\|_{(\infty)}^\# = \|T\|_{(n)}^\# = \max \left\{ \|T\|, \frac{1}{n} \|T\|_1 \right\} = \|T\|$. □

The following Lemma is Theorem 6.19 of [4].

Lemma 5.12. *Let \mathcal{N} be a type II_1 factor with the faithful normal tracial state $\tau_{\mathcal{N}}$ and $0 \leq t \leq 1$. Then*

$$(\|T\|_{(t), \mathcal{N}, \tau_{\mathcal{N}}}^{\#} = \max\{t\|T\|, \|T\|_{1, \tau_{\mathcal{N}}}\}, \quad \forall T \in \mathcal{M},$$

where $\|T\|_{(t), \mathcal{N}, \tau_{\mathcal{N}}}$ is the Ky Fan t -th norm of T with respect to $\tau_{\mathcal{N}}$ and $\|T\|_{1, \tau_{\mathcal{N}}} = \tau_{\mathcal{N}}(|T|)$.

Theorem 5.13. *Let \mathcal{M} be a type II_{∞} factor and $0 \leq t \leq \infty$. Then for all $T \in \mathcal{J}(\mathcal{M})$,*

$$\|T\|_{(t)}^{\#} = \begin{cases} \max\{t\|T\|, \|T\|_1\}, & \text{if } 0 \leq t \leq 1; \\ \max\{\|T\|, \frac{1}{t}\|T\|_1\}, & \text{if } 1 < t \leq \infty. \end{cases}$$

Proof. Let $T \in \mathcal{J}(\mathcal{M})$ and $0 < t < \infty$. There is a finite projection E in \mathcal{M} such that $T = ETE$ is in \mathcal{M}_E . We can assume that $\tau(E) = n > t$. Let $\tau_E(ESE) = \frac{\tau(ESE)}{\tau(E)}$. Then (\mathcal{M}_E, τ_E) is a type II_1 factor and τ_E is the unique tracial state on \mathcal{M}_E . If $0 < t \leq 1$, by Lemma 4.1,

$$\begin{aligned} t\|T\|_{(t)} &= \sup\{|\tau(UTE')| : U \in \mathcal{U}(\mathcal{M}_E), E' \in \mathcal{P}(\mathcal{M}_E), \tau(E') = t\} \\ &= \tau(E) \cdot \sup\{|\tau_E(UTE')| : U \in \mathcal{U}(\mathcal{M}_E), E' \in \mathcal{P}(\mathcal{M}_E), \tau_E(E') = t/n\} \\ &= \tau(E) \frac{t}{n} \|T\|_{(\frac{t}{n}), \mathcal{M}_E, \tau_E} = t\|T\|_{(\frac{t}{n}), \mathcal{M}_E, \tau_E}, \end{aligned}$$

where $\|T\|_{(\frac{t}{n}), \mathcal{M}_E, \tau_E}$ means the Ky Fan $\frac{t}{n}$ -th norm of $T \in \mathcal{M}_E$ with respect to the tracial state τ_E .

Hence, $\|T\|_{(t)} = \|T\|_{(\frac{t}{n}), \mathcal{M}_E, \tau_E}$. By Lemma 5.5 and Lemma 5.13,

$$\|T\|_{(t)}^{\#} = \tau(E) \cdot \left(\|T\|_{(\frac{t}{n}), \mathcal{M}_E, \tau_E}^{\#} \right) = n \max \left\{ \frac{t}{n} \|T\|, \|T\|_{1, \tau_E} \right\} = \max\{t\|T\|, \|T\|_1\}.$$

If $1 < t < \infty$, then $\|T\|_{(t)} = t\|T\|_{(\frac{t}{n}), \mathcal{M}_E, \tau_E}$. By Lemma 5.5 and Lemma 5.12,

$$\|T\|_{(t)}^{\#} = \tau(E) \cdot \left(t\|T\|_{(\frac{t}{n}), \mathcal{M}_E, \tau_E}^{\#} \right) = \frac{n}{t} \max \left\{ \frac{t}{n} \|T\|, \|T\|_{1, \tau_E} \right\} = \max \left\{ \|T\|, \frac{1}{t} \|T\|_1 \right\}.$$

Similar to the proof of Theorem 5.11, $\|T\|_{(\infty)}^{\#} = \|T\|$. □

5.3 Second dual norms

The following lemma follows from Theorem C of [4].

Lemma 5.14. *Let $(\mathcal{N}, \tau_{\mathcal{N}})$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $\|\cdot\|$ be a symmetric gauge norm on \mathcal{M} . Then $\|\cdot\|^{\#\#} = \|\cdot\|$.*

Theorem 5.15. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra satisfying the conditions A and B in section 3.4. If $\|\cdot\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then $\|\cdot\|^{\#}$ is also a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|^{\#\#} = \|\cdot\|$ on $\mathcal{J}(\mathcal{M})$.*

Proof. By Proposition 5.7, $\|\cdot\|^{\#}$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Furthermore, both $\|\cdot\|^{\#\#}$ and $\|\cdot\|$ are symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. We need to prove that $\|T\| = \|T\|^{\#\#}$ for every positive operator $T \in \mathcal{J}(\mathcal{M})$. Let E be a finite projection in \mathcal{M} such that $T \in \mathcal{M}_E$. By Lemma 5.5 and Lemma 5.14,

$$\begin{aligned} \|T\|_{\mathcal{M}, \tau}^{\#\#} &= \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}), \|X\|_{\mathcal{M}, \tau}^{\#} \leq 1\} \\ &= \sup\{\tau(E) \cdot |\tau_E(TX)| : X \in \mathcal{M}_E, \|X\|_{\mathcal{M}, \tau}^{\#} \leq 1\} \\ &= \sup\{|\tau_E(T(\tau(E)X))| : X \in \mathcal{M}_E, \|\tau(E)X\|_{\mathcal{M}_E, \tau_E}^{\#} \leq 1\} \\ &= \|T\|_{\mathcal{M}_E, \tau_E}^{\#\#} = \|T\|_{\mathcal{M}_E, \tau_E} = \|T\|. \end{aligned}$$

□

6 Main result

Throughout this section, we assume that (\mathcal{M}, τ) is a semi-finite von Neumann algebra.

Lemma 6.1. *Let $f(x) = \sum_{k=1}^n a_k \chi_{[\alpha_{k-1}, \alpha_k)}(x)$, where $a_1 \geq a_2 \geq \dots \geq a_n \geq 0 (= a_{n+1})$ and $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n < \infty$. For $T \in \mathcal{M}$, define*

$$\|T\|_f = \int_0^\infty f(s) \mu_s(T) ds.$$

Then

$$\|T\|_f = \sum_{k=1}^n \min\{\alpha_k, 1\} (a_k - a_{k+1}) \|T\|_{(\alpha_k)}.$$

Proof. Since $t\|T\|_{(t)} = \int_0^t \mu_s(T) ds$ for $0 \leq t \leq 1$ and $\|T\|_{(t)} = \int_0^t \mu_s(T) ds$ for $1 \leq t < \infty$, summation by parts shows that

$$\begin{aligned} \|T\|_f &= \int_0^\infty f(s) \mu_s(T) ds = a_1 \int_0^{\alpha_1} \mu_s(T) ds + a_2 \int_{\alpha_1}^{\alpha_2} \mu_s(T) ds + \dots + a_n \int_{\alpha_{n-1}}^{\alpha_n} \mu_s(T) ds \\ &= \sum_{k=1}^n \min\{\alpha_k, 1\} (a_k - a_{k+1}) \|T\|_{(\alpha_k)}. \end{aligned}$$

□

Corollary 6.2. $\|\cdot\|_f$ is a symmetric gauge norm associated to \mathcal{M} and therefore a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Furthermore, if $\tau(E) = 1$ then $\|E\|_f = \int_0^1 f(x)dx$.

Lemma 6.3. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra and $E \in \mathcal{M}$ be a (non-zero) finite projection. Suppose \mathcal{M}_E is a diffuse von Neumann algebra and $T, X \in \mathcal{M}_E$ are positive operators such that $T = a_1 E_1 + \cdots + a_n E_n$, $E_1 + \cdots + E_n = E$, and $\tau(E_1) = \cdots = \tau(E_n)$. Then there is a sequence of simple positive operators $X_n \in \mathcal{M}_E$ satisfying the following conditions:

1. $0 \leq X_1 \leq X_2 \leq \cdots \leq X$ and hence $0 \leq \mu_s(X_1) \leq \mu_s(X_2) \leq \cdots \leq \mu_s(X)$ for all $s \in [0, \infty)$;
2. $\lim_{n \rightarrow \infty} \mu_s(X_n) = \mu_s(X)$ for almost all $s \in [0, \infty)$;
3. there exists an $r_n \in \mathbf{N}$ such that $T = a_{n,1} E_{n,1} + \cdots + a_{n,r_n} E_{n,r_n}$ and $X_n = b_{n,1} F_{n,1} + \cdots + b_{n,r_n} F_{n,r_n}$, where $E_{n,1} + \cdots + E_{n,r_n} = F_{n,1} + \cdots + F_{n,r_n} = E$ and $\tau(E_{n,i}) = \tau(F_{n,j})$ for $1 \leq i, j \leq r_n$.

Proof. Since \mathcal{M}_E is diffuse, there is a separable diffuse abelian von Neumann subalgebra \mathcal{A} of \mathcal{M}_E such that $X \in \mathcal{A}$. Let θ be a $*$ -isomorphism from \mathcal{A} onto $L^\infty[0, 1]$ such that $\tau_E = \int_0^1 dx \cdot \theta$. Let $f(x) = \theta(X)$. We can choose a sequence of simple functions $f_n(x)$ in $L^\infty[0, 1]$ such that $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f(x)$ and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for almost all x . Let $X_n = \theta^{-1}(f_n(x))$. Then $X_n \in \mathcal{M}_E$ and $0 \leq X_1 \leq X_2 \leq \cdots \leq X$. By Lemma 2.6,

$$\begin{aligned} \mu_s(T) &= \inf\{\|TF\| : F \in \mathcal{P}(\mathcal{M}), \quad \tau(F^\perp) = s\} \\ &= \inf\{\|TF\| : F \in \mathcal{P}(\mathcal{M}_E), \quad \tau_E(F^\perp) = s\tau(E)\} = f^*(\tau(E)s), \end{aligned}$$

where $f^*(x)$ is the non-increasing rearrangement of $f(x)$. Therefore, we obtain 1 and 2. To obtain 3, we need only construct $f_n(x) = \alpha_{n,1} \chi_{I_{n,1}}(x) + \cdots + \alpha_{n,r_n} \chi_{I_{n,r_n}}(x)$ such that $m(I_{n,1}) = \cdots = m(I_{n,r_n}) = \frac{\tau_E(E_1)}{k_n}$, for some $k_n \in \mathbf{N}$. □

Let \mathcal{F} be the set of non-increasing, non-negative, right continuous simple functions $f(x)$ on $[0, \infty)$ with compact supports such that $\int_0^1 f(x)dx \leq 1$. $\forall f(x) \in \mathcal{F}$, $f(x) = \sum_{k=1}^n a_k \chi_{[\alpha_{k-1}, \alpha_k)}(x)$, where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 (= a_{n+1})$ and $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \infty$.

Recall that a normalized norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ of a semi-finite von Neumann algebra \mathcal{M} is a norm on $\mathcal{J}(\mathcal{M})$ such that $\|E\| = 1$ for some projection E with $\tau(E) = 1$. The following theorem is the main result of this paper.

Theorem 6.4. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra satisfying the conditions **A** and **B** in section 3.4. If $\|\cdot\|$ is a normalized symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then there is a subset \mathcal{F}' of \mathcal{F} containing the characteristic function on $[0, 1]$ such that for all $T \in \mathcal{J}(\mathcal{M})$, $\|T\| = \sup\{\|T\|_f : f \in \mathcal{F}'\}$.*

Proof. Suppose $\|\cdot\|$ is a normalized symmetric gauge norm on \mathcal{M} . Let $\mathcal{F}' = \{\mu_s(X) : X \text{ is a simple positive operator in } \mathcal{J}(\mathcal{M}), \|X\|^\# \leq 1\}$. For every positive operator $X \in \mathcal{J}(\mathcal{M})$ such that $\|X\|^\# \leq 1$, by Proposition 4.6, $\int_0^1 \mu_s(X) ds = \|X\|_{(1)} \leq \|X\|^\# \leq 1$. If E is a projection such that $\tau(E) = 1$, then $\|E\|^\# = 1$ by Proposition 5.7. Note that $\mu_s(E) = \chi_{[0,1]}(s)$. Therefore, $\mathcal{F}' \subset \mathcal{F}$ and $\chi_{[0,1]}(s) \in \mathcal{F}'$. For $T \in \mathcal{J}(\mathcal{M})$, define

$$\|T\|' = \sup\{\|T\|_f : f \in \mathcal{F}'\}.$$

By Corollary 6.2, $\|\cdot\|'$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. By Lemma 3.21, to prove that $\|\cdot\|' = \|\cdot\|$, we need to prove $\|T\|' = \|T\|$ for every positive simple operator $T \in \mathcal{J}(\mathcal{M})$ such that $T = a_1 E_1 + \cdots + a_n E_n$ and $\tau(E_1) = \cdots = \tau(E_n) = c > 0$.

By Lemma 5.9 and Theorem 5.15,

$$\|T\| = \sup \left\{ c \sum_{k=1}^n a_k b_k : X = b_1 E_1 + \cdots + b_n E_n \geq 0, \|X\|^\# \leq 1 \right\}.$$

Note that if $X = b_1 E_1 + \cdots + b_n E_n$ is a simple positive operator in $\mathcal{J}(\mathcal{M})$ and $\|X\|^\# \leq 1$, then $\mu_s(X) \in \mathcal{F}'$ and $\|T\|_{\mu_s(X)} = \int_0^\infty \mu_s(X) \mu_s(T) ds = c \sum_{k=1}^n a_k^* b_k^*$, where $\{a_k^*\}$ and $\{b_k^*\}$ are nondecreasing rearrangements of $\{a_k\}$ and $\{b_k\}$, respectively. By the Hardy-Littlewood-Polya Theorem [10], $\sum_{k=1}^n a_k b_k \leq \sum_{k=1}^n a_k^* b_k^*$. Hence,

$$\begin{aligned} \|T\| &= \sup \left\{ c \sum_{k=1}^n a_k b_k : X = b_1 E_1 + \cdots + b_n E_n \geq 0, \|X\|^\# \leq 1 \right\} \\ &\leq \sup\{\|T\|_f : f \in \mathcal{F}'\} = \|T\|'. \end{aligned}$$

Now we need to prove that $\|T\|' \leq \|T\|$. Let $X \in \mathcal{J}(\mathcal{M})$ be a positive simple operator such that $\|X\|^\# \leq 1$. We need to prove that $\|T\|_{\mu_s(X)} \leq \|T\|$. Since $T, X \in \mathcal{J}(\mathcal{M})$, there is a finite projection $E \in \mathcal{M}$ such that $T, X \in \mathcal{M}_E$.

First, we assume that $T = \tilde{a}_1 \tilde{E}_1 + \cdots + \tilde{a}_r \tilde{E}_r$ and $X = b_1 F_1 + \cdots + b_r F_r$, where $E_1 + \cdots + E_r = F_1 + \cdots + F_r = E$, $\tau(\tilde{E}_i) = \tau(F_j) = \tilde{c}$ for $1 \leq i, j \leq r$, $\tilde{a}_1 \geq \cdots \geq \tilde{a}_r$ and $b_1 \geq \cdots \geq b_r$. Let $Y = b_1 \tilde{E}_1 + \cdots + b_r \tilde{E}_r$. Then $\mu_s(Y) = \mu_s(X)$. Since $\tau(\tilde{E}_i) = \tau(F_j) = \tilde{c}$ for $1 \leq i, j \leq r$ and $\text{Aut}(\mathcal{M}, \tau)$ acts on \mathcal{M} ergodically, there is a $\theta \in \text{Aut}(\mathcal{M}, \tau)$ such that $\theta(\tilde{E}_i) = F_i$ for $1 \leq i \leq r$. Hence $\theta(Y) = X$. Since $\|\cdot\|^\#$ is a symmetric gauge norm,

$\|Y\|^\# = \|X\|^\# \leq 1$. By Corollary 5.4, $\|T\| \geq \tau(TY) = \tilde{c} \sum_{k=1}^r \tilde{a}_k b_k = \int_0^\infty \mu_s(Y) \mu_s(T) ds = \int_0^\infty \mu_s(X) \mu_s(T) ds = \|T\|_{\mu_s(X)}$.

Now we consider the general case. Since (\mathcal{M}_E, τ_E) satisfies the weak Dixmier property, by Lemma 3.16, \mathcal{M}_E is either a finite dimensional von Neumann algebra such that $\tau(F) = \tau(F')$ for every two minimal projections F and F' or \mathcal{M}_E is a diffuse von Neumann algebra. The first case is proved. If \mathcal{M}_E is a diffuse von Neumann algebra, by Lemma 6.3, we can construct a sequence of simple positive operators $X_n \in \mathcal{J}(\mathcal{M})$ satisfying the following conditions:

1. $0 \leq X_1 \leq X_2 \leq \cdots \leq X$ and hence $0 \leq \mu_s(X_1) \leq \mu_s(X_2) \leq \cdots \leq \mu_s(X)$ for all $s \in [0, \infty)$;
2. $\lim_{n \rightarrow \infty} \mu_s(X_n) = \mu_s(X)$ for almost all $s \in [0, \infty)$;
3. there exists an $r_n \in \mathbf{N}$ such that $T = a_{n,1}E_{n,1} + \cdots + a_{n,r_n}E_{n,r_n}$ and $X = b_{n,1}F_{n,1} + \cdots + b_{n,r_n}F_{n,r_n}$, where $E_{n,1} + \cdots + E_{n,r_n} = F_{n,1} + \cdots + F_{n,r_n} = E$ and $\tau(E_{n,i}) = \tau(F_{n,j})$ for $1 \leq i, j \leq r_n$.

By 1 and Corollary 3.4, $\|X_n\|^\# \leq \|X\| \leq 1$ for all $n = 1, 2, \dots$. By 3 and the above arguments, for every n , $\|T\| \geq \|T\|_{\mu_s(X_n)}$. By 1, 2 and the Monotone Convergence theorem,

$$\|T\|_{\mu_s(X)} = \int_0^\infty \mu_s(X) \mu_s(T) ds = \lim_{n \rightarrow \infty} \int_0^\infty \mu_s(X_n) \mu_s(T) ds = \lim_{n \rightarrow \infty} \|T\|_{\mu_s(X_n)} \leq \|T\|.$$

□

Corollary 6.5. *Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra as in Theorem 6.4 and let $\|\cdot\|$ be a normalized symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Then $\|\cdot\|$ can be extended to a normalized symmetric gauge norm $\|\cdot\|'$ associated to \mathcal{M} .*

Proof. For $T \in \mathcal{M}$, define $\|T\|' = \max\{\|T\|_f : f \in \mathcal{F}'\}$. Then $\|\cdot\|'$ is an extension of $\|\cdot\|$. □

Remark 6.6. In Corollary 6.5, the extension is not unique. Indeed, define $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$ by $\|T\| = \|T\|$ if T is a finite rank operator and $\|T\| = \infty$ if T is an infinite rank operator. It is easy to see that $\|\cdot\|$ defines a unitarily invariant norm associated to $\mathcal{B}(\mathcal{H})$ such that the restriction of $\|\cdot\|$ on $\mathcal{J}(\mathcal{H})$ is the operator norm.

7 Unitarily invariant norms related to semi-finite factors

As the first application of Theorem 6.4, we set up a structure theorem for unitarily invariant norms related to semi-finite factors. Recall that \mathcal{F} is the set of non-increasing, non-negative, right continuous simple functions $f(x)$ on $[0, \infty)$ with compact supports such that $\int_0^1 f(x)dx \leq 1$.

Theorem 7.1. *Let \mathcal{M} be a semi-factor and let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. Then there is a subset \mathcal{F}' of \mathcal{F} containing the characteristic function on $[0, 1]$ such that for all $T \in \mathcal{J}(\mathcal{M})$, $\|T\| = \sup\{\|T\|_f : f \in \mathcal{F}'\}$.*

Proof. Combining Theorem 3.29 and Proposition 3.22, we prove the theorem. \square

The next corollary also follows from Theorem 6.4.

Corollary 7.2. *Let $\|\cdot\|$ be a normalized symmetric gauge norm on $\mathcal{J}(L^\infty[0, \infty))$. Then there is a subset \mathcal{F}' of \mathcal{F} containing the characteristic function on $[0, 1]$ such that for all $T \in \mathcal{J}(L^\infty[0, \infty))$,*

$$\|T\| = \sup\{\|T\|_f : f \in \mathcal{F}'\}.$$

Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \text{Tr}$. By the proof of Theorem 6.4, if $f \in \mathcal{F}'$, then $f(s) = \mu_s(X)$ for some finite rank operator $X \in \mathcal{B}(\mathcal{H})$, $X \geq 0$ and $\|X\|^\# \leq 1$. Write $\mu_s(X) = s_1(X)\chi_{[0,1)}(s) + s_2(X)\chi_{[1,2)}(s) + \dots$, where $s_1(X), s_2(X), \dots$, are s -numbers of X . Since $\int_0^1 \mu_s(X) \leq 1$, $s_1(X) \leq 1$. By Lemma 6.1 and simple computations, for every $T \in \mathcal{J}(\mathcal{H})$,

$$\|T\|_{\mu_s(X)} = s_1(X)s_1(T) + s_2(X)s_2(T) + \dots,$$

where $s_1(T), s_2(T), \dots$, are s -numbers of T .

Let $\mathcal{G} = \{(a_1, a_2, \dots) : 1 \geq a_1 \geq a_2 \geq \dots \geq 0 \text{ and } a_n = 0 \text{ except for finitely many terms}\}$. For $(a_1, a_2, \dots) \in \mathcal{G}$ and $T \in \mathcal{J}(\mathcal{H})$, define

$$\|T\|_{(a_1, a_2, \dots)} = a_1 s_1(T) + a_2 s_2(T) + \dots. \quad (3)$$

Then $\|T\|_{(a_1, a_2, \dots)} = \|T\|_f$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$, where $f(x) = a_1 \chi_{[0,1)}(x) + a_2 \chi_{[1,2)}(x) + \dots$. By identifying $\mu_s(X)$ with $(s_1(X), s_2(X), \dots)$ in \mathcal{G} , we obtain the following corollary.

Corollary 7.3. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$. Then there is a subset \mathcal{G}' of \mathcal{G} , $(1, 0, \dots) \in \mathcal{G}'$, such that for all $T \in \mathcal{J}(\mathcal{H})$,*

$$\|T\| = \sup\{a_1 s_1(T) + a_2 s_2(T) + \dots : (a_1, a_2, \dots) \in \mathcal{G}'\},$$

where $s_1(T), s_2(T), \dots$ are s -numbers of T .

Similar to the proof of Corollary 7.3, we have the following corollary.

Corollary 7.4. *Let $\|\cdot\|$ be a normalized symmetric gauge norm on $c_{00} = \mathcal{J}(l^\infty(\mathbb{N}))$. Then there is a subset \mathcal{G}' of \mathcal{G} , $(1, 0, \dots) \in \mathcal{G}'$, such that for all $(x_1, x_2, \dots) \in c_{00}$,*

$$\|(x_1, x_2, \dots)\| = \sup\{a_1 x_1^* + a_2 x_2^* + \dots : (a_1, a_2, \dots) \in \mathcal{G}'\},$$

where (x_1^*, x_2^*, \dots) is the nonincreasing rearrangement of $(|x_1|, |x_2|, \dots)$.

8 Unitarily invariant norms and symmetric gauge norms

Lemma 8.1. *Let θ_1, θ_2 be two embeddings from $(L^\infty[0, \infty), \int_0^\infty dx)$ into a type II_∞ factor (\mathcal{M}, τ) . If $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$, then $\|\theta_1(f)\| = \|\theta_2(f)\|$ for every positive function $f \in \mathcal{J}(L^\infty[0, \infty))$.*

Proof. For $f \in \mathcal{J}(L^\infty[0, \infty))$, let $\|f\|_1 = \|\theta_1(f)\|$ and $\|f\|_2 = \|\theta_2(f)\|$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are gauge norms on $\mathcal{J}(L^\infty[0, \infty))$. By Lemma 3.21, to prove $\|\cdot\|_1 = \|\cdot\|_2$ on $\mathcal{J}(L^\infty[0, \infty))$, we need to prove $\|f\|_1 = \|f\|_2$ for every simple function $f(x)$ in $\mathcal{J}(L^\infty[0, \infty))$. If $f(x) \in \mathcal{J}(L^\infty[0, \infty))$ is a simple function, then there is a unitary operator U in \mathcal{M} such that $U\theta_1(f)U^* = \theta_2(f)$. Hence $\|f\|_1 = \|f\|_2$. \square

The following theorem generalizes von Neumann's classical result [28] on unitarily invariant norms on $M_n(\mathbb{C})$.

Theorem 8.2. *There is a one-to-one correspondence between*

1. *unitarily invariant norms on $M_n(\mathbb{C})$ and symmetric gauge norms on \mathbb{C}^n ,*
2. *unitarily invariant norms on a type II_1 factor and symmetric gauge norms on $L^\infty[0, 1]$,*
3. *unitarily invariant norms on $\mathcal{J}(\mathcal{H})$ and symmetric gauge norms on $c_{00} = \mathcal{J}(l^\infty(\mathbb{N}))$,*
4. *unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ of a type II_∞ factor \mathcal{M} and symmetric gauge norms on $\mathcal{J}(L^\infty[0, \infty))$,*

respectively. Precisely, let \mathcal{M} be a semi-finite factor and \mathcal{A} be the corresponding abelian von Neumann algebra as the above:

- if $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ and θ is an embedding from \mathcal{A} into \mathcal{M} , then the restriction of $\|\cdot\|$ to $\mathcal{J}(\theta(\mathcal{A}))$ defines a symmetric gauge norm on $\mathcal{J}(\mathcal{A})$;
- conversely, if $\|\cdot\|'$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{A})$ and $T \in \mathcal{J}(\mathcal{M})$, then $\|\mu_s(T)\|'$ defines a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$, where $\mu_s(T)$ is the classical s -numbers of T if $\mathcal{M} = M_n(\mathbb{C})$ or $\mathcal{M} = \mathcal{B}(\mathcal{H})$, and $\mu_s(T)$ is defined as in [4] if \mathcal{M} is a type II_1 factor.

Proof. We refer to [4] for the proof of case 1 and case 2. We only prove the theorem for case 4. The proof of case 3 is similar. We may assume that both norms on $\mathcal{J}(\mathcal{M})$ and $\mathcal{J}(L^\infty[0, \infty))$ are normalized. By the definition of Ky Fan norms, there is a one-to-one correspondence between Ky Fan t -th norms on $\mathcal{J}(\mathcal{M})$ and Ky Fan t -th norms on $\mathcal{J}(L^\infty[0, \infty))$ as in Theorem 8.2. By Theorem 7.1 and Corollary 7.2, there is a one-to-one correspondence between normalized unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ and normalized symmetric gauge norms on $\mathcal{J}(L^\infty[0, \infty))$ as in the theorem. \square

Example 8.3. For $1 \leq p \leq \infty$, the L^p -norm on $(L^\infty[0, \infty), \int_0^\infty dx)$ defined by

$$\|f(x)\|_p = \begin{cases} \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}, & 1 \leq p < \infty; \\ \text{essup } |f|, & p = \infty \end{cases}$$

is a normalized symmetric gauge norm on $(L^\infty[0, \infty), \int_0^\infty dx)$. By Theorem 8.2, the induced norm for $T \in \mathcal{J}(\mathcal{M})$ of a type II_∞ factor \mathcal{M} defined by

$$\|T\|_p = \begin{cases} = (\tau(|T|^p))^{1/p} = \left(\int_0^1 |\mu_s(T)|^p ds\right)^{1/p}, & 1 \leq p < \infty; \\ \|T\|, & p = \infty \end{cases}$$

is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. The norms $\{\|\cdot\|_p : 1 \leq p \leq \infty\}$ are called L^p -norms on $\mathcal{J}(\mathcal{M})$.

Example 8.4. For $1 \leq p \leq \infty$, the l^p -norm defined on $\mathcal{J}(l^\infty(\mathbb{N}))$ by

$$\|(x_1, x_2, \dots)\|_p = \begin{cases} (|x_1|^p + |x_2|^p + \dots)^{1/p}, & 1 \leq p < \infty; \\ \text{essup } \{|x_n| : n = 1, 2, \dots\}, & p = \infty \end{cases}$$

is a normalized symmetric gauge norm on $\mathcal{J}(l^\infty(\mathbb{N}))$. By Theorem 8.2, the induced norm for T in $\mathcal{J}(\mathcal{H})$ defined by

$$\|T\|_p = \begin{cases} = (\tau(|T|^p))^{1/p} = (s_1(T)^p + s_2(T)^p + \dots)^{1/p}, & 1 \leq p < \infty; \\ \|T\|, & p = \infty \end{cases}$$

is a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$. The norms $\{\|\cdot\|_p : 1 \leq p \leq \infty\}$ are called L^p -norms on $\mathcal{J}(\mathcal{H})$.

Theorem 8.2 establishes the one to one correspondence between unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ of an infinite semi-finite factor \mathcal{M} and symmetric gauge norms on $\mathcal{J}(\mathcal{A})$ of an abelian von Neumann algebra \mathcal{A} . The following theorem further establishes the one to one correspondence between the dual norms on $\mathcal{J}(\mathcal{M})$ and the dual norms on $\mathcal{J}(\mathcal{A})$, which plays a key role in the studying of duality and reflexivity of the completion of $\mathcal{J}(\mathcal{M})$ with respect to unitarily invariant norms.

Theorem 8.5. *Let \mathcal{M} be a type II_∞ factor (or $\mathcal{B}(\mathcal{H})$). If $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\cdot\|_1$ on $\mathcal{J}(L^\infty[0, \infty))$ (or $\mathcal{J}(l^\infty(\mathbf{N}))$) respectively as in Theorem 8.2, then $\|\cdot\|^\#$ is the unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\cdot\|_1^\#$ on $\mathcal{J}(L^\infty[0, \infty))$ (or $\mathcal{J}(l^\infty(\mathbf{N}))$) respectively as in Theorem 8.2.*

Proof. We only prove the theorem for type II_∞ factors. The case of type I_∞ factors is similar. Let $\|\cdot\|_2$ be the unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\cdot\|_1^\#$ on $\mathcal{J}(L^\infty[0, \infty))$ as in Theorem 8.2. By Lemma 3.21, to prove $\|\cdot\|_2 = \|\cdot\|^\#$ on $\mathcal{J}(\mathcal{M})$, we need to prove $\|T\|_2 = \|T\|^\#$ for every simple positive operator $T = a_1 E_1 + \cdots + a_n E_n$ in $\mathcal{J}(\mathcal{M})$ such that $\tau(E_1) = \cdots = \tau(E_n) = c$. We may assume that $a_1 \geq \cdots \geq a_n \geq 0$. Then $\mu_s(T) = a_1 \chi_{[0, c)}(s) + \cdots + a_n \chi_{[(n-1)c, nc)}(s)$. By Lemma 5.9,

$$\|T\|^\# = \sup \left\{ c \sum_{k=1}^n a_k b_k : X = b_1 E_1 + \cdots + b_n E_n \geq 0, \|X\| \leq 1 \right\}.$$

By the Hardy-Littlewood-Polya Theorem,

$$\|T\|^\# = \sup \left\{ c \sum_{k=1}^n a_k b_k : X = b_1 E_1 + \cdots + b_n E_n \geq 0, b_1 \geq \cdots \geq b_n \geq 0, \|X\| \leq 1 \right\}.$$

By Theorem 8.2 and Lemma 5.9,

$$\|T\|_2 = \|\mu_s(T)\|^\# = \sup \left\{ c \sum_{k=1}^n a_k b_k : g(s) = b_1 \chi_{[0, c)}(s) + \cdots + b_n \chi_{[(n-1)c, nc)}(s) \geq 0, \|g(s)\| \leq 1 \right\}.$$

By the Hardy-Littlewood-Polya Theorem,

$$\begin{aligned} \|T\|_2 = \|\mu_s(T)\|^\# &= \sup \left\{ c \sum_{k=1}^n a_k b_k : g(s) = b_1 \chi_{[0, c)}(s) + \cdots + b_n \chi_{[(n-1)c, nc)}(s) \geq 0, \right. \\ &\quad \left. b_1 \geq \cdots \geq b_n \geq 0, \|g(s)\| \leq 1 \right\}. \end{aligned}$$

Note that if $b_1 \geq \cdots \geq b_n \geq 0$, then $\mu_s(b_1 E_1 + \cdots + b_n E_n) = b_1 \chi_{[0, c)}(s) + \cdots + b_n \chi_{[(n-1)c, nc)}(s)$. Since $\|\cdot\|$ is the unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\cdot\|_1$ on $\mathcal{J}(L^\infty[0, \infty))$ as in Theorem 8.2, $\|b_1 E_1 + \cdots + b_n E_n\| \leq 1$ if and only if $\|b_1 \chi_{[0, c)}(s) + \cdots + b_n \chi_{[(n-1)c, nc)}(s)\|_1 \leq 1$. Therefore, $\|T\|_2 = \|T\|^\#$. \square

Example 8.6. If $p = 1$, let $q = \infty$. If $1 < p < \infty$, let $q = \frac{p}{p-1}$. Then the L^q norm on $\mathcal{J}(L^\infty[0, \infty))$ defined by Example 8.3 is the dual norm of the L^p -norm on $\mathcal{J}(L^\infty[0, \infty))$. By Theorem 8.5, the L^q -norm on $\mathcal{J}(\mathcal{M})$ of a type II_∞ factor \mathcal{M} is the dual norm of the L^p -norm on $\mathcal{J}(\mathcal{M})$.

Example 8.7. If $p = 1$, let $q = \infty$. If $1 < p < \infty$, let $q = \frac{p}{p-1}$. Then the l^q -norm on $\mathcal{J}(l^\infty(\mathbf{N}))$ defined by Example 8.4 is the dual norm of the l^p -norm on $\mathcal{J}(l^\infty(\mathbf{N}))$. By Theorem 8.5, the L^q -norm on $\mathcal{J}(\mathcal{H})$ is the dual norm of the L^p -norm on $\mathcal{J}(\mathcal{H})$.

9 Ky Fan's dominance theorem

The following theorem generalizes Ky Fan's dominance theorem.

Theorem 9.1. *Let \mathcal{M} be a semi-finite factor and $S, T \in \mathcal{J}(\mathcal{M})$. If $\|S\|_{(t)} \leq \|T\|_{(t)}$ for all Ky Fan t -th norms, $0 \leq t \leq \infty$, then $\|S\| \leq \|T\|$ for all unitarily invariant norms $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$.*

Proof. Let $\|\cdot\|$ be a unitarily invariant norm on \mathcal{M} . By Lemma 6.1, $\|S\|_f \leq \|T\|_f$ for every $f \in \mathcal{F}$. By Theorem 7.1, $\|S\| \leq \|T\|$. \square

Corollary 9.2. *Let $S, T \in \mathcal{J}(\mathcal{H})$. If $\|S\|_{(n)} \leq \|T\|_{(n)}$ for all Ky Fan n -th norms, $n = 1, 2, \dots$, then $\|S\| \leq \|T\|$ for all unitarily invariant norms $\|\cdot\|$ on $\mathcal{J}(\mathcal{H})$.*

10 Completion of $\mathcal{J}(\mathcal{M})$ with respect to unitarily invariant norms

In this section, we assume that \mathcal{M} is an infinite semi-finite factor with a tracial weight τ and $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. Recall that $\mathcal{J}(\mathcal{M})$ is the set of operators T in \mathcal{M} such that $T = ETE$ for some finite projection $E \in \mathcal{M}$. The completion of $\mathcal{J}(\mathcal{M})$ with respect to $\|\cdot\|$ is denoted by $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$. For $1 \leq p < \infty$, we will use the traditional notation $L^p(\mathcal{M}, \tau)$ to denote the completion of $\mathcal{J}(\mathcal{M})$ with respect to the L^p -norm defined as in Example 8.3 or Example 8.4. We will denote by $\mathcal{K}(\mathcal{M})$ the completion of $\mathcal{J}(\mathcal{M})$ with respect to the operator norm. Let $\widetilde{\mathcal{M}}$ be the completion of \mathcal{M} in the measure-topology in the sense of Nelson [18]. Recall that a neighborhood $N(\epsilon, \delta)$ of $0 \in \mathcal{M}$ in the measure topology (see [18]) is

$$N(\epsilon, \delta) = \{T \in \mathcal{M}, \text{ there is a projection } E \in \mathcal{M} \text{ such that } \tau(E) < \delta \text{ and } \|TE^\perp\| < \epsilon\}$$

10.1 $L^1(\mathcal{M}, \tau)$

In this section, we summarize some classical results on $\widetilde{\mathcal{M}}$ and $L^1(\mathcal{M}, \tau)$ that will be useful.

Theorem 10.1. (Nelson, [18]) *$\widetilde{\mathcal{M}}$ is a $*$ -algebra and every element in $\widetilde{\mathcal{M}}$ is a closed, densely defined operator affiliated with \mathcal{M} .*

In the following, we define s -numbers for unbounded operators in $\widetilde{\mathcal{M}}$ as [6].

Definition 10.2. For $T \in \widetilde{\mathcal{M}}$ and $0 \leq s < \infty$, define the s -numbers of T by

$$\mu_s(T) = \inf\{\|TE\| : E \in \mathcal{M} \text{ is a projection such that } \tau(E^\perp) = s\}.$$

Theorem 10.3. (Fack and Kosaki, [6]) *Let T and T_n be a sequence of operators in $\widetilde{\mathcal{M}}$ such that $\lim_{n \rightarrow \infty} T_n = T$ in the measure-topology. Then for almost all $s \in [0, \infty)$, $\lim_{n \rightarrow \infty} \mu_s(T_n) = \mu_s(T)$.*

Let $\{T_n\}$ be a sequence of operators in $\mathcal{J}(\mathcal{M})$ such that $T = \lim_{n \rightarrow \infty} T_n$ in the L^1 -norm. By Lemma 4.2, $\{\tau(T_n)\}$ is a Cauchy sequence in \mathbb{C} . Define $\tau(T) = \lim_{n \rightarrow \infty} \tau(T_n)$. It is obvious that $\tau(T)$ does not depend on the sequence $\{T_n\}$. In this way, τ is extended to a linear functional on $L^1(\mathcal{M}, \tau)$. The following lemma is due to Nelson [18].

Lemma 10.4. *Let \mathcal{M} be a type II_∞ factor with a faithful normal tracial weight τ . Then there is a natural continuous injective map from $L^1(\mathcal{M}, \tau)$ into $\widetilde{\mathcal{M}}$. Furthermore, if $\{T_n\} \subset \mathcal{J}(\mathcal{M})$ is a Cauchy sequence in the L^1 -norm and $\lim_{n \rightarrow \infty} T_n = T$ in the measure topology, then $T \in L^1(\mathcal{M}, \tau)$ and $\lim_{n \rightarrow \infty} T_n = T$ in the L^1 -norm.*

10.2 Embedding of $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ into $\widetilde{\mathcal{M}}$

In this subsection, we will prove that $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ can be embedded into $\widetilde{\mathcal{M}}$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the embedding is very simple due to the following proposition.

Proposition 10.5. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$. Then $\overline{\mathcal{J}(\mathcal{H})_{\|\cdot\|}}$ is a selfadjoint two-sided ideal of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{K}_1(\mathcal{H}) \subseteq \overline{\mathcal{J}(\mathcal{H})_{\|\cdot\|}} \subseteq \mathcal{K}(\mathcal{H})$.*

Proof. By Corollary 4.8, we obtain the proposition. □

Lemma 10.6. *Let \mathcal{M} be a type II_∞ factor and $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. There is a natural continuous map Φ from $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ to $\widetilde{\mathcal{M}}$ that extends the identity map from $\mathcal{J}(\mathcal{M})$ to $\mathcal{J}(\mathcal{M})$.*

Proof. We may assume that $\|\cdot\|$ is a normalized unitarily invariant norm, i.e., $\|E\| = 1$ if $\tau(E) = 1$. If $\{T_n\}$ in $\mathcal{J}(\mathcal{M})$ is a Cauchy sequence with respect to $\|\cdot\|$, then $\{T_n\}$ in $\mathcal{J}(\mathcal{M})$ is a Cauchy sequence with respect to $\|\cdot\|_{(1)}$ by Proposition 4.6. By Corollary 4.4, for any $\delta > 0$ and $T \in \mathcal{J}(\mathcal{M})$ such that $\|T\|_{(1)} < \delta$, $\tau(\chi_{(\delta,\infty)}(|T|)) \leq \frac{\|T\|_{(1)}}{\delta}$. Hence, if $\{T_n\}$ is a Cauchy sequence in \mathcal{M} with respect to the $\|T\|_{(1)}$ norm, then $\{T_n\}$ is a Cauchy sequence in the measure topology. So there is a natural map Φ from $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ that extends the identity map from $\mathcal{J}(\mathcal{M})$ to $\mathcal{J}(\mathcal{M})$. \square

Lemma 10.7. *Let $\{T_n\}$ be a sequence of operators in $\mathcal{J}(\mathcal{M})$ such that $\lim_{n \rightarrow \infty} T_n = T$ with respect to $\|\cdot\|$ and $X \in \mathcal{J}(\mathcal{M})$. Then $\lim_{n \rightarrow \infty} T_n X = \Phi(T)X$ in the L^1 -norm.*

Proof. By Lemma 10.6, $\lim_{n \rightarrow \infty} T_n = \Phi(T)$ in the measure topology. Hence $\lim_{n \rightarrow \infty} T_n X = \Phi(T)X$ in the measure topology (see Theorem 1 of [18]). By Corollary 5.4, $\|T_n X - T_m X\|_1 \leq \|T_n - T_m\| \cdot \|X\|^\#$. Hence $\{T_n X\}$ is a Cauchy sequence in the L^1 -norm. By Lemma 10.4, $\lim_{n \rightarrow \infty} T_n X = \Phi(T)X$ in the L^1 -norm. \square

Corollary 10.8. *Let $\{T_n\}$ be a sequence of operators in $\mathcal{J}(\mathcal{M})$ such that $\lim_{n \rightarrow \infty} T_n = T$ with respect to $\|\cdot\|$ and $X \in \mathcal{J}(\mathcal{M})$. Then $\lim_{n \rightarrow \infty} \tau((T_n - \Phi(T))X) = 0$.*

Lemma 10.9. *For all $T \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$,*

$$\|T\| = \sup\{|\tau(\Phi(T)X)| : X \in \mathcal{J}(\mathcal{M}), \|X\|^\# \leq 1\}.$$

Proof. Let $\{T_n\}$ be a sequence of operators in $\mathcal{J}(\mathcal{M})$ such that $\lim_{n \rightarrow \infty} T_n = T$ with respect to $\|\cdot\|$. By Corollary 10.8 and Lemma 5.3, for every $X \in \mathcal{J}(\mathcal{M})$ such that $\|X\|^\# \leq 1$,

$$|\tau(\Phi(T)X)| = \lim_{n \rightarrow \infty} |\tau(T_n X)| \leq \lim_{n \rightarrow \infty} \|T_n\| = \|T\|.$$

Hence, $\|T\| \geq \sup\{|\tau(\Phi(T)X)| : X \in \mathcal{J}(\mathcal{M}), \|X\|^\# \leq 1\}$.

We need to prove that $\|T\| \leq \sup\{|\tau(\Phi(T)X)| : X \in \mathcal{J}(\mathcal{M}), \|X\|^\# \leq 1\}$. Let $\epsilon > 0$. Since $\lim_{n \rightarrow \infty} T_n = T$ with respect to $\|\cdot\|$, there exists an N such that $\|T - T_N\| < \epsilon/3$. For T_N , there is an $X \in \mathcal{J}(\mathcal{M})$, $\|X\|^\# \leq 1$, such that $\|T_N\| \leq |\tau(T_N X)| + \epsilon/3$. By Corollary 10.8 and Corollary 5.4,

$$|\tau((T_N - \Phi(T))X)| = \lim_{n \rightarrow \infty} |\tau((T_N - T_n)X)| \leq \lim_{n \rightarrow \infty} \|T_N - T_n\| \cdot \|X\|^\# \leq \|T_N - T\| < \epsilon/3.$$

So $|\tau(\Phi(T)X)| \geq |\tau(T_N X)| - |\tau((T_N - \Phi(T))X)| \geq \|T_N\| - \epsilon/3 - \epsilon/3 \geq \|T\| - \epsilon$. Therefore, $\|T\| \leq \sup\{|\tau(\Phi(T)X)| : X \in \mathcal{J}(\mathcal{M}), \|X\|^\# \leq 1\}$. \square

Proposition 10.10. *Let \mathcal{M} be a type II_∞ factor and $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. There is a natural continuous injective map Φ from $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ to $\widetilde{\mathcal{M}}$ that extends the identity map from $\mathcal{J}(\mathcal{M})$ to $\mathcal{J}(\mathcal{M})$.*

Proof. By Lemma 10.6, we need only to prove that Φ is injective. To prove Φ is injective, we need to prove that if $\{T_n\}$ in $\mathcal{J}(\mathcal{M})$ is a Cauchy sequence with respect to $\|\cdot\|$ and $T_n \rightarrow 0$ in the measure topology, then $\lim_{n \rightarrow \infty} \|T_n\| = 0$. Suppose $T = \lim_{n \rightarrow \infty} T_n$ with respect to the norm $\|\cdot\|$. Then $\Phi(T) = 0$. By Lemma 10.9,

$$\|T\| = \sup\{|\tau(\Phi(T)X)| : X \in \mathcal{J}(\mathcal{M}), \|X\|^\# \leq 1\} = 0.$$

Hence, $T = 0$. □

By Proposition 10.10, we will identify $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ with $\Phi(T)$ and consider $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ as a subset of $\widetilde{\mathcal{M}}$.

Corollary 10.11. $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ is a linear subspace of $\widetilde{\mathcal{M}}$ satisfying the following conditions:

1. if $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$, then $T^* \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$;
2. $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ if and only if $|T| \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$;
3. if $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ and $A, B \in \mathcal{J}(\mathcal{M})$, then $ATB \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ and $\|ATB\| \leq \|A\| \cdot \|T\| \cdot \|B\|$.

In particular, $\|\cdot\|$ can be extended to a unitarily invariant norm, also denoted by $\|\cdot\|$, on $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$.

10.3 Elements in $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$

The following theorem generalizes Theorem 6.4. Its proof is based on Lemma 10.9 and is similar to the proof of Theorem 6.4. So we omit the proof. Recall that \mathcal{F} is the set of non-increasing, non-negative, right continuous simple functions $f(x)$ on $[0, \infty)$ with compact supports such that $\int_0^1 f(x)dx \leq 1$.

Theorem 10.12. If $\|\cdot\|$ is a normalized unitarily invariant norm on an infinite semi-finite factor $\mathcal{J}(\mathcal{M})$, then there is a subset \mathcal{F}' of \mathcal{F} containing the characteristic function on $[0, 1]$ such that for all $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$,

$$\|T\| = \sup\{\|T\|_f : f \in \mathcal{F}'\}.$$

Combining Theorem 10.12 and Theorem 10.3, we have the following corollaries.

Corollary 10.13. Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$ and $\|\cdot\|'$ be the corresponding symmetric gauge norm on $c_{00} = \mathcal{J}(l^\infty(\mathbf{N}, \tau))$. If $T \in \mathcal{K}(\mathcal{H})$, then $T \in \overline{\mathcal{J}(\mathcal{H})_{\|\cdot\|}}$ if and only if $(\mu_1(T), \mu_2(T), \dots)$ is in the completion of c_{00} with respect to $\|\cdot\|'$. In this case, $\|T\| = \|\mu_s(T)\|'$.

Corollary 10.14. *Let \mathcal{M} be a type II_∞ factor and $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|'$ be the corresponding symmetric gauge norm on $\mathcal{J}(L^\infty[0, \infty))$. If $T \in \widetilde{\mathcal{M}}$, then $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ if and only if $\mu_s(T) \in \overline{\mathcal{J}(L^\infty[0, \infty))_{\|\cdot\|'}}$. In this case, $\|T\| = \|\mu_s(T)\|'$.*

Example 10.15. Let $T \in \mathcal{K}(\mathcal{H})$ and $1 \leq p \leq \infty$. Then $T \in L^p(\mathcal{H}, \tau)$ if and only if $(\mu_1(T), \mu_2(T), \dots) \in l^p(\mathbf{N}, \tau)$. In this case, $\|T\|_p = (s_1(T)^p + s_2(T)^p + \dots)^{1/p}$.

Example 10.16. Let $T \in \widetilde{\mathcal{M}}$ and $1 \leq p \leq \infty$. Then $T \in L^p(\mathcal{M}, \tau)$ if and only if $\mu_s(T) \in L^p[0, \infty)$. In this case, $\|T\|_p = \left(\int_0^1 \mu_s(T)^p ds \right)^{1/p} = \left(\int_0^\infty \lambda^p d\mu_{|T|} \right)^{1/p}$.

10.4 A generalization of Hölder's inequality

The following theorem is a generalization of Hölder's inequality.

Theorem 10.17. *Let \mathcal{M} be an infinite semi-finite factor and $\|\cdot\|$ be a normalized unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. If $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ and $S \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|^\#}}$, then $TS \in L^1(\mathcal{M}, \tau)$ and $\|TS\|_1 \leq \|T\| \cdot \|S\|^\#$.*

Proof. By the polar decomposition and Corollary 10.11, we may assume that S and T are positive operators. Let T_n and S_n be two sequences of operators in $\mathcal{J}(\mathcal{M})$ such that $\lim_{n \rightarrow \infty} \|T - T_n\| = \lim_{n \rightarrow \infty} \|S - S_n\|^\# = 0$. Let K be a positive number such that $\|T_n\| \leq K$ and $\|S_n\|^\# \leq K$ for all n and $\epsilon > 0$. Then there is an N such that for all $m > n \geq N$, $\|T_m - T_n\| < \epsilon/(2K)$ and $\|S_m - S_n\|^\# < \epsilon/(2K)$. By Corollary 5.4, $\|T_m S_m - T_n S_n\|_1 \leq \|(T_m - T_n)S_m\|_1 + \|T_n(S_m - S_n)\|_1 \leq \|T_m - T_n\| \cdot \|S_m\|^\# + \|T_n\| \cdot \|S_m - S_n\|^\# < \epsilon$. This implies that $\{T_n S_n\}$ is a Cauchy sequence in \mathcal{M} in the L^1 -norm. Since $\lim_{n \rightarrow \infty} T_n S_n = TS$ in the measure topology, $\lim_{n \rightarrow \infty} T_n S_n = TS$ in the L^1 -norm by Lemma 10.7. By Corollary 5.4, $\|T_n S_n\|_1 \leq \|T_n\| \cdot \|S_n\|^\#$ for every n . Hence, $\|TS\|_1 \leq \|T\| \cdot \|S\|^\#$. \square

Combining Example 8.3, 8.4, 8.6, 8.7 and Theorem 10.17, we obtain the classical non-commutative Hölder's inequality.

Corollary 10.18. *Let \mathcal{M} be an infinite semi-finite factor with the faithful normal tracial weight τ . If $T \in L^p(\mathcal{M}, \tau)$ and $S \in L^q(\mathcal{M}, \tau)$, then $TS \in L^1(\mathcal{M}, \tau)$ and*

$$\|TS\|_1 \leq \|T\|_p \cdot \|S\|_q,$$

where $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

10.5 Some approximation results

Lemma 10.19. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{J}(\mathcal{H})$. If S, S_1, S_2, \dots , are bounded operators in $\mathcal{B}(\mathcal{H})$ such that $S = \lim_{n \rightarrow \infty} S_n$ in the strong operator topology, then*

$$\lim_{n \rightarrow \infty} \|S_n T - S T\| = 0.$$

Proof. Since $T \in \mathcal{J}(\mathcal{H})$, there is a finite rank projection E such that $T = ETE$. Since $S_n \rightarrow S$ in the strong operator topology, $S_n E \rightarrow SE$ in the operator topology. By Proposition 3.6,

$$\|S_n T - S T\| = \|S_n E T - S E T\| \leq \|S_n E - S E\| \cdot \|T\| \rightarrow 0.$$

□

Theorem 10.20. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{B}(\mathcal{H})$ and $T \in \overline{\mathcal{J}(\mathcal{H})_{\|\cdot\|}}$. If S, S_1, S_2, \dots , are bounded operators in $\mathcal{B}(\mathcal{H})$ such that $S = \lim_{n \rightarrow \infty} S_n$ in the strong operator topology, then*

$$\lim_{n \rightarrow \infty} \|S_n T - S T\| = 0.$$

Proof. Since $S_n \rightarrow S$ in the strong operator topology, there is a number $M > 0$ such that $\|S\| \leq M$ and $\|S_n\| \leq M$ for $n = 1, 2, \dots$. Let $\epsilon > 0$. Since $T \in \overline{\mathcal{J}(\mathcal{H})_{\|\cdot\|}}$, there is a $T' \in \mathcal{J}(\mathcal{H})$ such that $\|T' - T\| < \frac{\epsilon}{3M}$. By Lemma 10.19, there is an N such that $\|S_n T' - S T'\| < \frac{\epsilon}{3}$ for all $n \geq N$. By Proposition 3.6,

$$\begin{aligned} \|S_n T - S T\| &\leq \|S_n T - S_n T'\| + \|S_n T' - S T'\| + \|S T' - S T\| \\ &< \|S_n\| \cdot \|T - T'\| + \|S_n T' - S T'\| + \|S\| \cdot \|T' - T\| \\ &= \epsilon. \end{aligned}$$

□

Corollary 10.21. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$ and $T \in \overline{\mathcal{J}(\mathcal{H})_{\|\cdot\|}}$. If E_1, E_2, \dots , are finite rank projections in $\mathcal{B}(\mathcal{H})$ such that $\lim_{n \rightarrow \infty} E_n = I$ in the strong operator topology, then*

$$\lim_{n \rightarrow \infty} \|E_n T E_n - T\| = 0.$$

Let \mathcal{M} be a diffuse semi-finite von Neumann algebra. Then the set of finite projections $\{E_\alpha\}$ of \mathcal{M} is a partially ordered set such that $\lim_\alpha E_\alpha = I$ in the strong operator topology. The following lemma will be useful in the next section.

Lemma 10.22. *Let \mathcal{M} be a semi-finite von Neumann algebra, and $\|\cdot\|$ be a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$ and $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$. Then*

$$\lim_\alpha \|E_\alpha T E_\alpha - T\| = 0.$$

Proof. Apply the technique used in the proof of Theorem 10.20, we need only prove the lemma under the assumption $T \in \mathcal{J}(\mathcal{M})$. In this case, the lemma is obvious since there is a finite projection E such that $ETE = T$. □

11 Duality and Reflexivity of noncommutative Banach spaces

In this section, we assume that \mathcal{M} is an infinite semi-finite factor with a faithful normal tracial weight τ , $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|^\#$ is the dual unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. We consider the following two questions in this section:

Question 1: When $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|^\#}$ is the dual space of $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ in the following sense: for every $\phi \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}^\#$, there is a unique $X \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|^\#}$ such that

$$\phi(T) = \tau(TX), \quad \forall T \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$$

and $\|\phi\| = \|T\|$?

Question 2: When $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ is a reflexive space?

For a projection E in \mathcal{M} , we denote by $\overline{(\mathcal{M}_E)}_{\|\cdot\|}$ the completion of $\mathcal{M}_E = E\mathcal{M}E$ with respect to $\|\cdot\|$.

Lemma 11.1. *If $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|^\#}$ is the dual space of $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ in the sense of question 1, then $\overline{(\mathcal{M}_E)}_{\|\cdot\|^\#}$ is the dual space of $\overline{(\mathcal{M}_E)}_{\|\cdot\|}$ for every projection E in \mathcal{M} in the sense of question 1.*

Proof. Let $\phi \in \overline{(\mathcal{M}_E)}_{\|\cdot\|}^\#$. We can identify $\overline{(\mathcal{M}_E)}_{\|\cdot\|}$ with a Banach subspace of $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$. By the Hahn-Banach extension theorem, ϕ can be extended to a linear functional $\psi \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}^\#$. By the assumption of the lemma, there is an operator $X \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|^\#}$ such that for every operator $T \in \mathcal{J}(\mathcal{M})$, $\psi(T) = \tau(TX)$. There is a sequence of operators $X_n \in \mathcal{J}(\mathcal{M})$ such that $\lim_{n \rightarrow \infty} \|X_n - X\|^\# = 0$. Let $Y_n = EX_nE$ and $Y = EXE$. Then $Y_n \in \mathcal{M}_E$. By Proposition 3.22,

$$\lim_{n \rightarrow \infty} \|Y - Y_n\|^\# \leq \lim_{n \rightarrow \infty} \|E\| \cdot \|X - X_n\|^\# \cdot \|E\| = 0.$$

Hence, $Y \in \overline{(\mathcal{M}_E)}_{\|\cdot\|^\#}$. For every $T \in \mathcal{M}_E$, $\phi(T) = \psi(T) = \tau(TX) = \tau(ETEX) = \tau(TY)$. This implies that $\overline{(\mathcal{M}_E)}_{\|\cdot\|^\#}$ is the dual space of $\overline{(\mathcal{M}_E)}_{\|\cdot\|}$ for every projection E in \mathcal{M} . \square

Let E be a (non-zero) finite projection in \mathcal{M} . Recall that $\mathcal{M}_E = E\mathcal{M}E$ is a finite von Neumann algebra with a faithful normal tracial state τ_E such that $\tau_E(T) = \frac{\tau(T)}{\tau(E)}$ for $T \in \mathcal{M}_E$. If $\|\cdot\|$ is a norm on \mathcal{M}_E , the dual norm of $T \in \mathcal{M}_E$ with respect to τ_E is defined by

$$\|T\|_{\mathcal{M}_E, \tau_E}^\# = \sup\{|\tau_E(TX)| : X \in \mathcal{M}_E, \|X\| \leq 1\}.$$

In the following, we denote by $\overline{(\mathcal{M}_E)_{\|\cdot\|_{\mathcal{M}_E, \tau_E}^\#}}$ the completion of \mathcal{M}_E with respect to $\|\cdot\|_{\mathcal{M}_E, \tau_E}^\#$. By Lemma 5.5, $\|T\|^\# = \tau(E)\|T\|_{\mathcal{M}_E, \tau_E}^\#$ for every $T \in \mathcal{M}_E$. Hence, $\overline{(\mathcal{M}_E)_{\|\cdot\|^\#}} = \overline{(\mathcal{M}_E)_{\|\cdot\|_{\mathcal{M}_E, \tau_E}^\#}}$ as linear spaces.

The next lemma follows from Theorem **H** of [4].

Lemma 11.2. *Let \mathcal{N} be a type II_1 factor with a faithful normal tracial state $\tau_{\mathcal{N}}$. Let $\|\cdot\|$ be a unitarily invariant norm on \mathcal{N} and $\|\cdot\|^\#$ be the dual unitarily invariant norm on \mathcal{N} . Let $\|\cdot\|_1$ be the symmetric gauge norm on $(L^\infty[0, 1], \int_0^1 dx)$ corresponding to $\|\cdot\|$ on \mathcal{N} as in Theorem 8.5. Then the following conditions are equivalent:*

1. $\overline{\mathcal{N}_{\|\cdot\|^\#}}$ is the dual space of $\overline{\mathcal{N}_{\|\cdot\|}}$ in the sense of question 1;
2. $\overline{L^\infty[0, 1]_{\|\cdot\|_1^\#}}$ is the dual space of $\overline{L^\infty[0, 1]_{\|\cdot\|_1}}$ in the sense of question 1.

The next lemma follows from Corollary **1** of [4].

Lemma 11.3. *Let \mathcal{N} be a finite factor with a faithful normal tracial state $\tau_{\mathcal{N}}$, and let $\|\cdot\|$ be a unitarily invariant norm on \mathcal{N} . If \mathcal{A} is a separable abelian von Neumann subalgebra of \mathcal{N} and $\mathbf{E}_{\mathcal{A}}$ is the normal conditional expectation from \mathcal{N} onto \mathcal{A} preserving $\tau_{\mathcal{N}}$, then $\|\mathbf{E}_{\mathcal{A}}(T)\| \leq \|T\|$ for all $T \in \mathcal{N}$.*

Definition 11.4. Let (\mathcal{M}, τ) be a semi-finite von Neumann algebra. An abelian von Neumann subalgebra \mathcal{A} is called a *normal* abelian von Neumann subalgebra if τ is a semi-finite tracial weight on \mathcal{A} .

The abelian algebra consists of bounded diagonal operators on a Hilbert space \mathcal{H} is a normal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. However, the abelian von Neumann algebra generated by the bilateral shift operator is not a normal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. By [25], if \mathcal{A} is a normal abelian von Neumann subalgebra of \mathcal{M} , then there is a normal conditional expectation $\mathbf{E}_{\mathcal{A}}$ from \mathcal{M} onto \mathcal{A} preserving τ .

The following theorem is the main result of this section.

Theorem 11.5. *Let \mathcal{M} be a type II_∞ factor (or $\mathcal{B}(\mathcal{H})$), $\|\cdot\|$ be a unitarily invariant norm on \mathcal{M} and $\|\cdot\|^\#$ be the dual unitarily invariant norm on \mathcal{M} . Let $\|\cdot\|_1$ be the symmetric gauge norm on $\mathcal{J}(L^\infty[0, \infty))$ (or $\mathcal{J}(l^\infty(\mathbf{N}))$) respectively) corresponding to $\|\cdot\|$ on \mathcal{M} as in Theorem 8.2. Then the following conditions are equivalent:*

1. $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|^\#}}$ is the dual space of $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ in the following sense: $\forall \psi \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}^\#}$, there is a unique $X \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}^\#}$ such that $\psi(T) = \tau(TX)$ for all $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ and $\|\psi\| = \|X\|^\#$;

2. $\overline{\mathcal{J}(L^\infty[0, \infty))}_{\|\cdot\|_1^\#}$ (or $\overline{\mathcal{J}(l^\infty(\mathbf{N}))}_{\|\cdot\|_1^\#}$ respectively) is the dual space of $\overline{\mathcal{J}(L^\infty[0, \infty))}_{\|\cdot\|_1}$ (or $\overline{\mathcal{J}(l^\infty(\mathbf{N}))}_{\|\cdot\|_1}$ respectively) in the same sense.

Proof. We prove the theorem for type II_∞ factors. The proof for type I_∞ factors is similar.

“1 \Rightarrow 2”. By Theorem 8.2 and Lemma 2.2, there is a normal separable diffuse abelian von Neumann subalgebra \mathcal{A} of \mathcal{M} and a $*$ -isomorphism α from \mathcal{A} onto $L^\infty[0, \infty)$ such that $\tau = \int_0^\infty dx \cdot \alpha$ and $\|\alpha(T)\|_1 = \|T\|$ for every $T \in \mathcal{J}(\mathcal{A})$. By Theorem 8.5, $\|\alpha(T)\|_1^\# = \|T\|^\#$ for every $T \in \mathcal{J}(\mathcal{A})$. So we need only prove that $\overline{\mathcal{J}(\mathcal{A})}_{\|\cdot\|^\#}$ is the dual space of $\overline{\mathcal{J}(\mathcal{A})}_{\|\cdot\|}$ in the sense of question 1. Let $\phi \in \overline{\mathcal{J}(\mathcal{A})}_{\|\cdot\|}^\#$. By the Hahn-Banach extension theorem, ϕ can be extended to a bounded linear functional ψ on $\overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ such that $\|\psi\| = \|\phi\|$. By the assumption of the theorem, there is an operator $X \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}^\#$ such that $\psi(S) = \tau(SX)$ for all $S \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}$ and $\|\psi\| = \|X\|^\#$. There is a sequence of operators $X_n \in \mathcal{J}(\mathcal{M})$ such that $\lim_{n \rightarrow \infty} \|X - X_n\| = 0$. Let $\mathbf{E}_\mathcal{A}$ be the normal conditional expectation from \mathcal{M} onto \mathcal{A} such that $\tau(T) = \tau(\mathbf{E}_\mathcal{A}(T))$ for all $T \in \mathcal{J}(\mathcal{M})$. Let $Y_n = \mathbf{E}_\mathcal{A}(X_n)$. By Lemma 11.3, $\|Y_n\| \leq \|X_n\|$. Therefore, $\{Y_n\}$ is a Cauchy sequence in $\mathcal{J}(\mathcal{A})$ with respect to norm $\|\cdot\|$. Let $Y = \lim_{n \rightarrow \infty} Y_n$ in $\overline{\mathcal{J}(\mathcal{A})}_{\|\cdot\|}$. Then for every $S \in \mathcal{J}(\mathcal{A})$,

$$\phi(S) = \psi(S) = \tau(SX) = \lim_{n \rightarrow \infty} \tau(SX_n) = \lim_{n \rightarrow \infty} \tau(\mathbf{E}_\mathcal{A}(SX_n)) = \lim_{n \rightarrow \infty} \tau(SY_n) = \tau(SY).$$

By Lemma 10.9, $\|\phi\| = \|Y\|_\mathcal{A}^\#$.

“2 \Rightarrow 1”. Let $\psi \in \overline{\mathcal{J}(\mathcal{M})}_{\|\cdot\|}^\#$, and let \mathcal{A} be a normal separable diffuse abelian von Neumann subalgebra of \mathcal{M} and ϕ be the restriction of ψ to $\mathcal{J}(\mathcal{A})$. By Lemma 2.2, we can identify (\mathcal{A}, τ) with $(L^\infty[0, \infty), \int_0^\infty dx)$. By the assumption of the theorem, $\overline{\mathcal{J}(L^\infty[0, \infty))}_{\|\cdot\|_1^\#}$ is the dual space of $\overline{\mathcal{J}(L^\infty[0, \infty))}_{\|\cdot\|_1}$. Hence, $\overline{(\mathcal{A})}_{\|\cdot\|^\#}$ is the dual space of $\overline{(\mathcal{A})}_{\|\cdot\|}$. Therefore, there is an operator $Y \in \overline{(\mathcal{A})}_{\|\cdot\|^\#}$ such that $\phi(T) = \tau(TY)$ for all $T \in \mathcal{J}(\mathcal{A})$ and $\|\phi\|^\# = \|Y\|^\#$.

Let $E \in \mathcal{A}$ be a finite projection. By Lemma 11.1, $\overline{(\mathcal{A}_E)}_{\|\cdot\|_1^\#}$ is the dual space of $\overline{(\mathcal{A}_E)}_{\|\cdot\|_1}$. By Lemma 5.5 and Lemma 11.2, $\overline{(\mathcal{M}_E)}_{\|\cdot\|^\#}$ is the dual space of $\overline{(\mathcal{M}_E)}_{\|\cdot\|}$. So there is a unique operator $X_E \in \overline{(\mathcal{M}_E)}_{\|\cdot\|^\#}$ such that for all $T \in \mathcal{M}_E$, $\psi(T) = \tau(TX_E)$. Define $\psi_E(T) = \psi(T)$ for $T \in \overline{\mathcal{M}_E}_{\|\cdot\|}$ and $\phi_E(S) = \phi(S)$ for $S \in \overline{\mathcal{A}_E}_{\|\cdot\|}^\#$. Then $\phi_E(S) = \phi(S) = \tau(SY) = \tau(ESY) = \tau(SEYE)$. By Lemma 11.2, $\|X_E\|^\# = \|\psi_E\| = \|\phi_E\| = \|EYE\|^\#$.

Let E, F be two finite projections in \mathcal{A} . Then $\|X_E - X_F\|^\# = \|\psi_E - \psi_F\| = \|\phi_E - \phi_F\| = \|EYE - F Y F\|^\#$. By Lemma 10.22, $\lim_\alpha \|E_\alpha Y E_\alpha - Y\|^\# = 0$. So $\{X_{E_\alpha}\}$ is a Cauchy

sequence in $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}^\#}$. Let $X = \lim_\alpha E_\alpha X E_\alpha$. Then $X \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}^\#}$ and $\psi(T) = \tau(TX)$ for all $T \in \mathcal{M}_E$ and finite projections $E \in \mathcal{A}$.

Let \mathcal{B} be another normal separable diffuse abelian von Neumann subalgebra of \mathcal{M} . Similar arguments as above shows that there is an operator $Z \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}^\#}$ such that $\psi(T) = \tau(TZ)$ for all $T \in \mathcal{M}_{E'}$ and finite projections $E' \in \mathcal{B}$. Let $E \in \mathcal{A}$ and $E' \in \mathcal{B}$ be finite projections. Then $F = E \wedge E'$ is a finite projection and $\psi(T) = \tau(TFXF) = \tau(TFZF)$ for all $T \in \mathcal{M}_F$. Therefore, $FXF = FZF$. Since we can choose $E_n \in \mathcal{A}$ and $E'_n \in \mathcal{B}$ such that $F_n = E_n \wedge E'_n \rightarrow 1$ in the strong operator topology, $F_n X F_n = F_n Z F_n$ implies that $X = Z$ by Theorem 3 of [18].

Note that if $T \in \mathcal{J}(\mathcal{M})$ is a positive operator, then there is a normal separable diffuse abelian von Neumann subalgebra of \mathcal{M} that contains T . Therefore, $\psi(T) = \tau(TX)$ for all positive operators $T \in \mathcal{J}(\mathcal{M})$ and hence for all operators in $\mathcal{J}(\mathcal{M})$. Since $X \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}^\#}$, by Theorem 10.17, $\psi(T) = \tau(TX)$ for all $T \in \overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$. By Lemma 10.9, $\|\psi\| = \|X\|^\#$. \square

Corollary 11.6. *For $1 \leq p < \infty$, $L^q(\mathcal{M}, \tau)$ is the dual space of $L^p(\mathcal{M}, \tau)$ in the sense of Question 1, where $\frac{1}{p} + \frac{1}{q} = 1$.*

The following theorem is a corollary of Theorem 11.5.

Theorem 11.7. *Let \mathcal{M} be a type II_∞ factor (or $\mathcal{B}(\mathcal{H})$), $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|_1$ be the symmetric gauge norm on $\mathcal{J}(L^\infty[0, \infty))$ (or $\mathcal{J}(l^\infty(\mathbf{N}))$) respectively corresponding to $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ as in Theorem 8.2. Then the following conditions are equivalent:*

1. $\overline{\mathcal{J}(\mathcal{M})_{\|\cdot\|}}$ is a reflexive space;
2. $\overline{\mathcal{J}(L^\infty[0, \infty))_{\|\cdot\|_1}}$ (or $\mathcal{J}(l^\infty(\mathbf{N}))$) respectively is a reflexive space.

Corollary 11.8. *For $1 < p < \infty$, $L^p(\mathcal{M}, \tau)$ is a reflexive space.*

Example 11.9. By Theorem 5.12 and Theorem 11.7, for $0 \leq t \leq \infty$, $\overline{\mathcal{M}_{\|\cdot\|_{(t)}}}$ is not a reflexive space.

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